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MATHEMATICAL MODELS

II SERIES

BY

ARNOLD EMCH

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Math

PREFACE

The mathematical models listed and described below have been planned and designed in the mathematical laboratory of the University of Illinois since the publication[1] of the first series in 1921.

As before, they are the results of an effort to represent certain features of mathematical instruction and research by adequate models, mechanisms, or graphs.

Only such problems have been considered and investigated which have not found the same constructive treatment elsewhere and of which there are no models in the market.

For those interested in models as listed in series I and II, arrangements can be made with private firms for the manufacture and sale of duplicates.

For further information, especially concerning the procuring of duplicates, apply to Arnold Emch, Associate Professor of Mathematics, University of Illinois.

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[1]University of Illinois Bulletin, Vol. XVIII, No. 12, formally dated Nov. 22, 1920.

MATHEMATICAL MODELS

II SERIES

19. Peano Surface.[1]

In his *Calcolo differentiale e principi di calcolo integrale*, published in 1884, Peano gave the first rigorous treatment of the theory of maxima and minima of functions of several variables. The development of this theory may also be found on pp. 181-188, and in

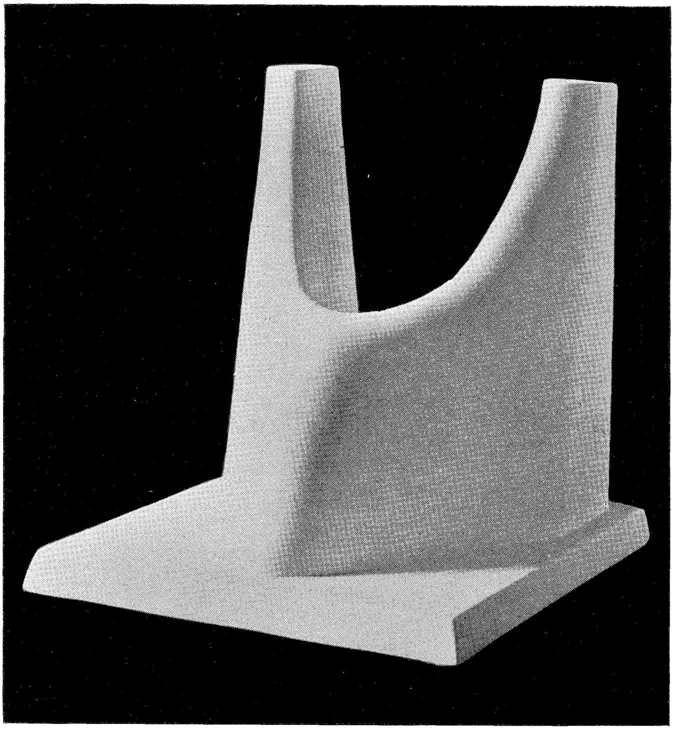

Fig. 1

note 133-136, on p. 332 of the German translation (1899). The note contains a discussion of the now famous function, representing a certain quartic surface which may properly be called *Peano Surface.*

[1]The last number of series I was 18. In this, as in prospective series, the models will be numbered continuously, so that in any inquiry concerning these models it will suffice to state the corresponding number.

By this example Peano demonstrated the falsity of some of the hitherto accepted criterions for maxima and minima of functions of several variables.

In most cases the errors are fundamentally due to the theorem, which, in general, is not true, that in Taylor's development of a function of several variables, the ratio of the remainder following a certain term of the expansion, to this term, approaches zero as a limit when the increments of the variables approach zero as a limit.

Peano's important criticisms are illustrated in a very conspicuous manner by the quartic surface with the Cartesian equation

$$z=\lambda\ (y^2-2px)\ (y^2-2qx).$$

The model, Fig. 1, represents the form of this Peano surface, when $\lambda=\ -{}^1/_{10}$, $p=2.5$, $q=0.5$.

Projective Description of Ruled Quartics

20. Ruled Quartic.

Generated by projectively related tangent planes of two cones, Fig. 2.

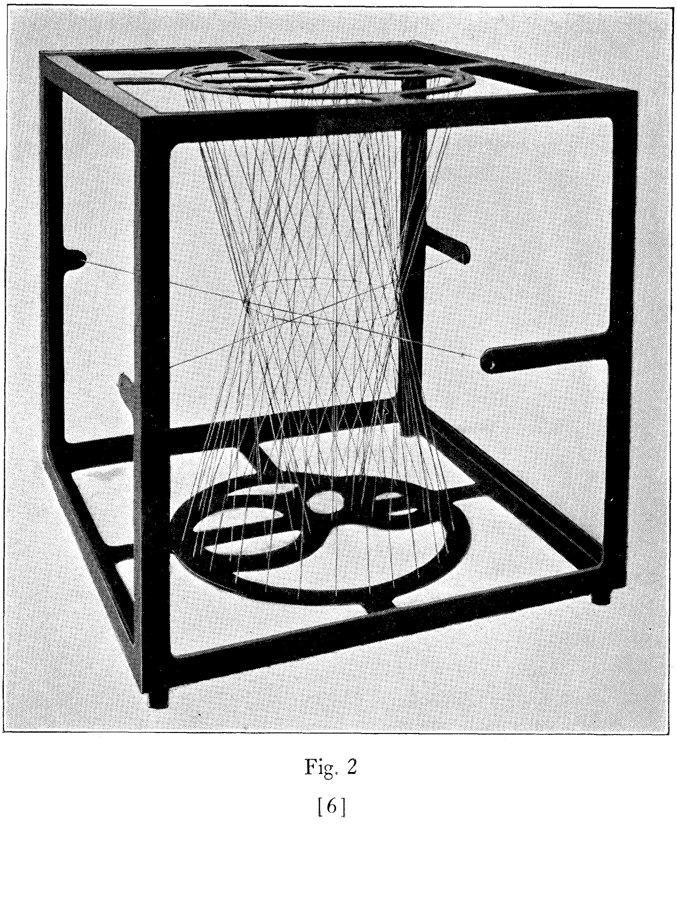

Fig. 2

Let the vertices of two right circular cones lie in the xy–plane at $(0, 0, 0)$ and $(a, 0, 0)$ and let the plane $z=1$ cut these cones in two circles with radii r_1 and r_2 respectively, so that the parametric representation of the first circle is

$$x=r_1\,\frac{1-t^2}{1+t^2}\,,\ y=r_1\,\frac{2t}{1+t^2}$$

Let the points of the second circle be projectively related to those of the first, in such a manner that the directions of corresponding radii are at right angles to each other, with angles Θ and $\Theta+\frac{\pi}{2}$, respectively.

The parametric equations of the second circle are therefore

$$x=a-r_2\,\frac{2t}{1+t^2}\,,\ y=r_2\,\frac{1-t^2}{1+t^2}$$

The tangent planes to the two cones at projectively corresponding points (generatrices) are easily established as

$$(x+r_1z)\,t^2-2yt-x+r_1z=0,$$
$$(y+r_2z)\,t^2+2\,(x-a)\,t-y+r_2z=0.$$

For every value of the parameter t the two planes intersect in a generatrix of a ruled surface whose equation is obtained by the elimination of t, and which may be transformed to the form:

$$[(r_1^2+r_2^2)(x^2+y^2)]z^2-(x^2+y^2+ax)^2=0.$$

The circle in the xy-plane through the vertices of the two cones is a nodal curve of the quartic. The z-axis is a nodal line.

21. Ruled Quartic.

Similar to No. 20, except that the angles of corresponding radii are Θ and $\frac{3\pi}{2}-\Theta$, Fig. 3.

The equation of this quartic is

$$[(r_1x+r_2y-ar_1)^2+(r_2x+r_1y)^2]z^2-(x^2-y^2-ax)^2=0.$$

The nodal curve consists of an equilateral hyperbola in the xy-plane, through the vertices of the cones.

The lines $r_1x+r_2y-ar_1=0$ and $r_2x+r_1y=0$ intersect in a point of $x^2-y^2-ax=0$, and the lines through this point, parallel to the $z-$axis is a nodal line.

The ruled quartic No. 9 of series I is of the same type as those described under Nos. 20 and 21. They are particular cases of the following kinematic method of generating a certain class of ruled surfaces. Given two right circular cones C_1 and C_2. A tangent plane e_1 revolves around C_1 with an angular velocity t_1. A tangent plane e_2 revolves around C_2 with an angular velocity t_2. When e_1 and e_2 are made to correspond to each other according to the law $\lambda t_1=\mu t_2$ where λ and μ are integers, then the locus of the line of intersection g of e_1 and e_2 is an algebraic ruled surface. In Nos 9, 20, 21 $\lambda=\mu$.

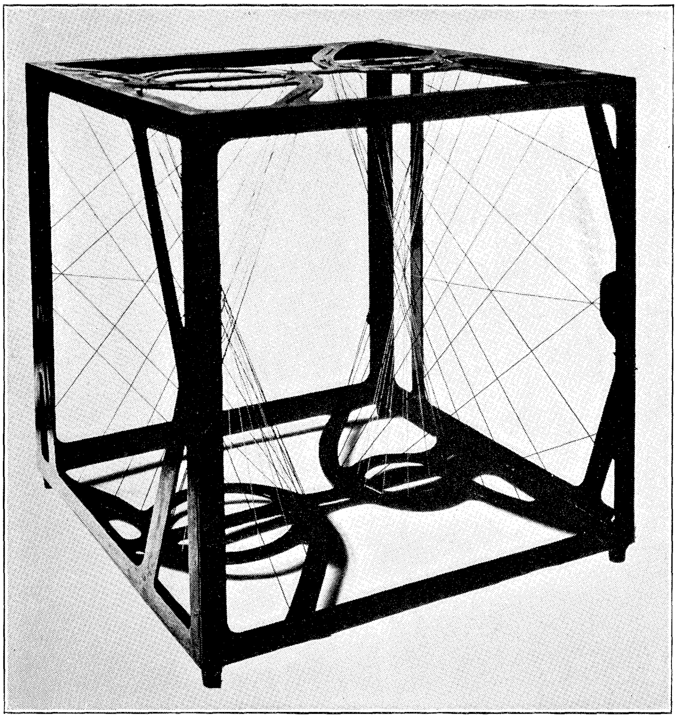

Fig. 3

Sextics

Considerable importance is attached to those sextics in space which may be obtained as intersections of quadric and cubic surfaces. Their projections are, in general, plane sextics of genus four, i.e., sextics with six double points.

Let Q and C represent a quadric and general cubic without a common line, or a common point of tangency. They intersect in a sextic S, which from an independent point O in space is projected on a plane in a plane sextic S'. To determine the double points of S', from O project S upon Q, so that the projected curve is of order 12 and contains S as a part. The proper projection is therefore also a sextic, say S^*. Now S^* cuts the cubic C in 18 points. Each of these points lies on Q and C, and lies therefore also on S. Among these 18 points those must be singled out which are obtained as the intersections of C with the conic K of contact of the tangent cone from O to Q. These are six in number, and they are projected from O upon the plane of projection as ordinary points of S'. Hence there are 12 points of intersection of S and S^* which may be grouped into six couples (A, A^*) with the following property: A^* which lies on S and S^* is obtained as the projection from O upon Q of a point A which lies on S. But since A^* also lies on S, it is obviously projected into the point A upon S^*. The projection of the couple (A, A^*) from O upon the plane of S' will therefore give a double point of S'. As there are six such couples, S', in general, has 6 double points and is of genus $10-6=4$.

When the quadric Q touches the cubic C in one, two, three, or four points, the number of double points will be increased accordingly, so that we get plane sextics S' with 7, 8, 9, 10 double points, or of genus 3, 2, 1, 0 respectively.

The models listed below show sextics of genus 4, drawn on transparent glass spheres.[1]

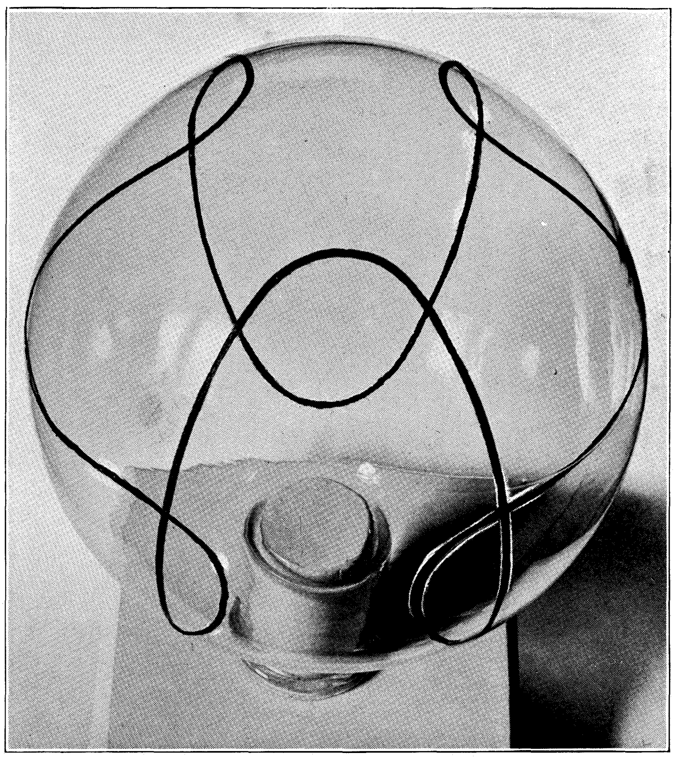

Fig. 4

[1]On the models the lines run smoothly and evenly. Refraction and reflexion of light mar the effect on the photographs and are the cause of irregularities.

22. Sextic.

Represents the intersection of a sphere and a cubic cone, Fig. 4. The equations for this model are

$$Q = x^2 + y^2 + z^2 - r^2 = 0$$
$$C = r^3 y(y^2 - 3x^2) - a^3 z^3 = 0$$

The orthographic projection S' of the space sextic S upon the xy-plane has the equation

$$r^6 y^2 (y - \sqrt{3}.x)^2 (y + \sqrt{3}.x)^2 + a^6 (x^2 + y^2 - r^2)^3 = 0$$

or

$$r^6 \rho^6 sin^2 3\theta - a^6 (r^2 - \rho^2)^3 = 0.$$

The lines $y = 0$, $y - \sqrt{3}.x = 0$, $y + \sqrt{3}.x = 0$ are cuspidal double tangents with the cusps on the circle $x^2 + y^2 = r^2$.

The sextic S is a closed non-singular curve on the sphere and its projection S' can easily be made to show 6 real double points.

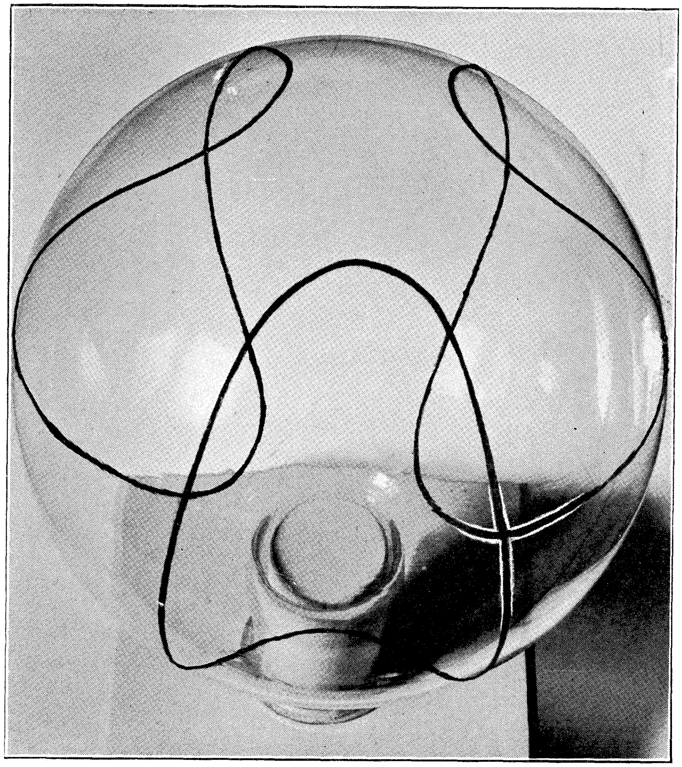

Fig. 5

23. Sextic.

Represents the intersection of a cubic with sphere, Fig. 5.

The equations are

$$Q = x^2 + y^2 + z^2 - r^2 = 0$$
$$C = y(y^2 - 3x^2) - kz^2 = 0.$$

The cubic is symmetric with respect to the z-plane and has the planes $y=0$, $y-\sqrt{3}.x=0$, $y+\sqrt{3}.x=0$ as tangent-planes along the lines where these planes cut the xy-plane.

The sextic S consists of 3 equal separate closed branches and its projection S' may also show all six double points as real. On the other hand S is seen to lie on four cubic cylinders.

This model also offers good opportunity for the exhibition of the problem of tritangent-planes of the sextic S, which is made possible by the drawing of the curve on a transparent surface.

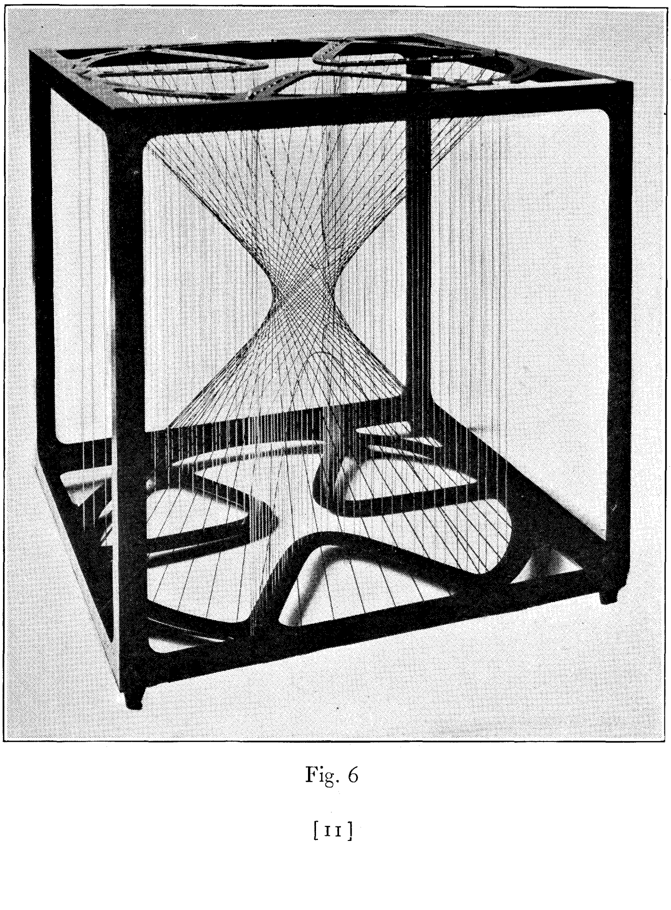

Fig. 6

24. Sextic.

String model of a sextic obtained as the intersection of a quadric and a cubic surface. The quadric is a hyperboloid of revolution of one sheet; the cubic a cylinder whose central axis coincides with that of the hyperboloid. The sextic consists of six branches which are distinct in the finite region of space, but which connect at infinity in a manner which can easily be ascertained by a study of the model, Fig. 6. The sextic is of genus 4.

25. Sextic.

This is also a string model and differs from No. 24 by the fact that the cubic is a tritangent-cylinder of the hyperboloid. The sextic thus acquires 3 effective double points, and projects into a plane sextic with 9 double points, i.e., into an elliptic sextic, Fig. 7.

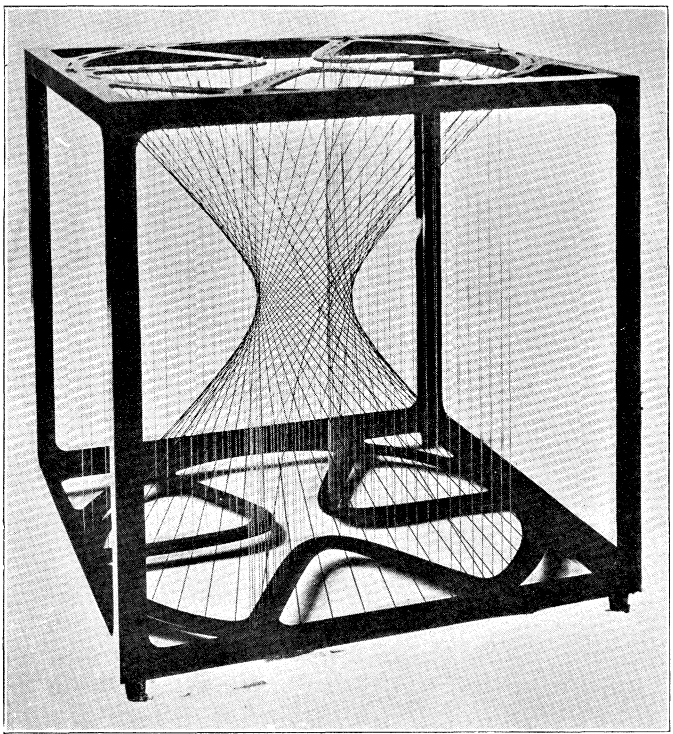

Fig. 7

Erratum. In Series I, No. 8, read second equation

$$x^2+y^2-(z-3a)^2-2\lambda x=0.$$