# SIMPLE APPLICATIONS
# OF
# TRIGONOMETRY TO ARTILLERY

BY
AUBREY J. KEMPNER, PH.D.

PRICE TWENTY-FIVE CENTS

PUBLISHED BY
THE UNIVERSITY OF ILLINOIS PRESS
URBANA, ILLINOIS

# SIMPLE APPLICATIONS

OF

# TRIGONOMETRY TO ARTILLERY

BY

AUBREY J. KEMPNER, PH.D.

Assistant Professor of Mathematics in the University of Illinois

PUBLISHED BY

THE UNIVERSITY OF ILLINOIS PRESS

NOVEMBER, 1918

## PREFACE

In writing this pamphlet it was the purpose of the author to bring some topics which occupy a prominent position in the standard textbooks on artillery into close connection with the mathematical work of a college course in trigonometry.

Mr. W. H. Rayner of the College of Engineering at the University of Illinois, who is at present giving a course in Orientation for Heavy Artillery. very kindly read the manuscript. The author is under obligation to Mr. Rayner for this assistance, since the range of his own knowledge of artillery matters is limited to a careful study of some of the standard textbooks.

The following books are particularly mentioned for reference:

1. *Alger, P. R.*, The Groundwork of Practical Naval Gunnery, 2nd. ed., 1917.

2. *Bishop, H. G.*, Elements of Modern Field Artillery, 2nd. ed., 1917.

3. *Moretti, O.* and *Danford, R. M.*, Notes on Training Field Artillery Details, 6th. ed., 1918.

4. *Spaulding, O. L., Jr.*, Notes on Field Artillery, 2nd. ed., 1917.

5. Gunnery and Explosives, War Department Document No. 391, 1911.

6. Manual of Field Artillery, Vol. 2, War Department Document No. 614, 1917.

The University Library possesses all of these books. In the text 1-4 will be referred to by the name of the author, 5 and 6 will be quoted as "Gunnery" and "Manual", respectively.

Besides, the author had the privilege of reading the proof-sheets of an article by Professor *J. K. Whittemore,* entitled "Firing Data," which has since appeared in the American Mathematical Monthly, October, 1918.

The problems 1 of page 7 and 1, 3, and 5 of pages 8, 9, and 10 are taken from Alger's excellent work; in the problems of Sections B and C emphasis has been laid on the character and on the degree of accuracy of the methods of approximation.

Sections B and C are independent of Section A.

## OUTLINE

# SECTION A.

## *Simple Applications of Trigonometry (without use of the "mil").* *The Trajectory.*

There is a marked difference between the methods of calculation applied in the Heavy or Coast Artillery and the Mobile or Field Artillery. In the Heavy Artillery angles must be determined much more accurately than in the Field Artillery, and formulae of approximation which are entirely sufficient for the latter service are totally inadequate to the needs of the former. In determining the firing data of a heavy gun, the mathematical operations required are often as delicate as in a refined experiment in physics, involving for example five place logarithms.

In particular, the approximations to which the use of the so-called "mil" measurement of angles leads, are not employed in the Heavy Artillery.

For this reason most problems of the present section are based on data referring to Heavy Artillery.

All definitions and therefore also all formulae hold without change for Field Artillery.

The projectile is assumed to move in vacuo; then the curve of flight, the *trajectory*, is part of a parabola; of course the actual path of the projectile is profoundly modified by the air-pressure. We mention particularly the following points concerning the actual path, the so-called "ballistic curve:"*

1. While the parabola has an axis of symmetry, the ballistic curve is not symmetric with respect to any line;

2. the ballistic curve lies entirely underneath the corresponding parabola;

3. the ballistic curve is more blunt at the end of the trajectory than at the beginning;

4. the highest point of flight lies in the second half of the curve;

5. for a considerable fraction of the whole path, the ballistic curve follows the corresponding parabola closely.

The great influence of the resistance of the air on the trajectory may be seen from the following little table which gives some interesting data for the three-inch Field Artillery gun.**

---

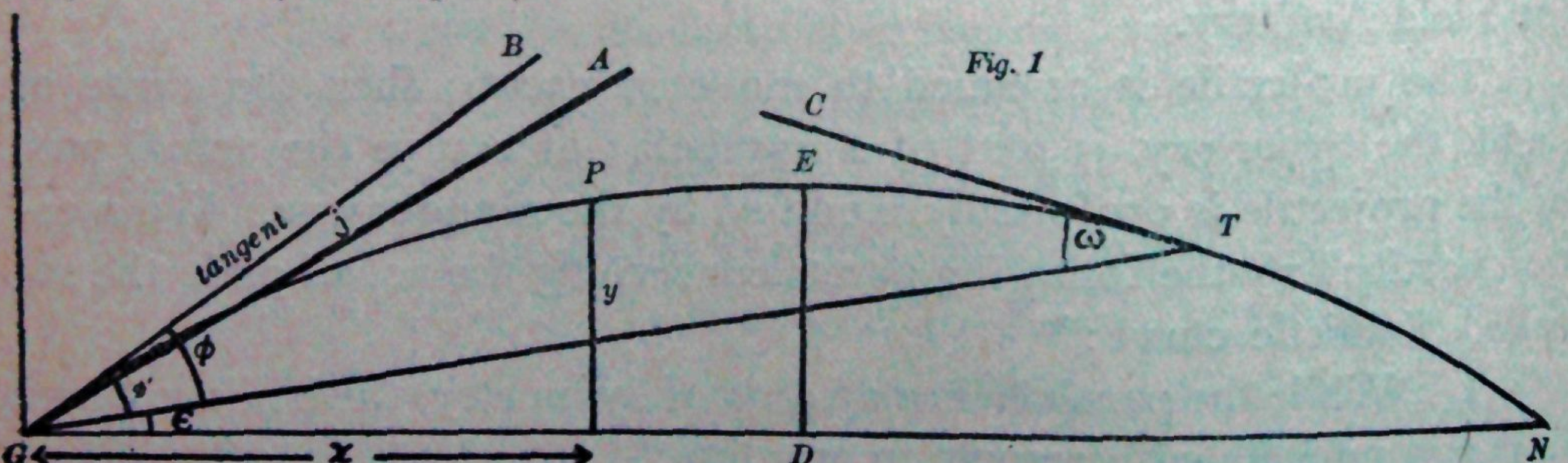
*Compare for this section: *Alger*, pp. 1-34; *Moretti* and *Danford*, Ch. III; Gunnery, pp. 13-24; Manual, pp. 97-101. There is lack of uniformity among authors concerning the *notation* of the elements defined on pp. 6, 7 of this pamphlet.

**Gunnery, p. 19.

| | Angle of Departure | Muzzle Velocity | Range | Maximum Ordinate | Time of Flight |
|---|---|---|---|---|---|
| Air | 1° 11.2′ | 1700 ft. sec. | 1000 yds. | 17.3 ft. | 2.07 sec. |
| Vacuo (appr.) | 1° 11.2′ | 1700 ft. sec. | 1245 yds. | 19.4 ft. | 2.20 sec. |
| Air | 2° 56.7′ | 1700 ft. sec. | 2000 yds. | 93.1 ft. | 4.46 sec. |
| Vacuo (appr.) | 2° 56.7′ | 1700 ft. sec. | 3089 yds. | 119.2 ft. | 4.75 sec. |
| Air | 5° 12′ | 1700 ft. sec. | 3000 yds. | 257.0 ft. | 7.83 sec. |
| Vacuo (appr.) | 5° 12′ | 1700 ft. sec. | 5434 yds. | 370.9 ft. | 9.63 sec. |
| Air | 7° 54.2′ | 1700 ft. sec. | 4000 yds. | 536.0 ft. | 11.25 sec. |
| Vacuo (appr.) | 7° 54.2′ | 1700 ft. sec. | 8200 yds. | 853.8 ft. | 14.61 sec. |
| Air | 11° 10.1′ | 1700 ft. sec. | 5000 yds. | 975.0 ft. | 15.12 sec. |
| Vacuo (appr.) | 11° 10.1′ | 1700 ft. sec. | 11440 yds. | 1694.0 ft. | 20.58 sec. |

The table brings out clearly the great flatness of the trajectory at ordinary ranges. For a rough construction of the ballistic curve, the abscissa corresponding to the maximum ordinate may be assumed three-fifths of the horizontal range. (See "Definitions," below).

*Definitions* (see Fig. 1):




Let $G$ in Fig. 1 be the gun (more accurately the muzzle of the gun), $T$ the target, $GN$ a horizontal line, $GB$ a tangent to the curve at $G$; then

Curve $GTN$ = trajectory (in vacuo a parabola, in air the "ballistic curve"),

$GN$ = horizontal range,

$GT$ = range (as a length); also line of sight; line of position,

$\angle NGT = \epsilon$, angle of site, angle of position,

$GB$ = line of departure,

$\angle TGB = \phi$, angle of departure,

$\angle NGB = \phi + \epsilon = \psi$, quadrant angle of departure,*

$\angle TGA = \phi'$, angle of elevation,

$\angle NGA = \phi' + \epsilon$, quadrant angle of elevation,*

*When no confusion is possible, the "quadrant angle of departure (of elevation)" is called simply the "angle of departure (of elevation)."

$\angle AGB = j$, jump,
$\angle CTG = \omega$, angle of fall,
$DE =$ maximum ordinate.

The meaning of most of these terms is clear from the figure. A few words must be said concerning the angles $\phi$, $\phi'$, $j$, $\epsilon$.

The angle $\phi'$ (or $\phi' + \epsilon$ when referred to the horizontal) is the angle which the axis of the bore makes with the line of sight (or with the horizontal) at the instant before firing. However, the axis of the gun changes its direction by a small (experimentally known) angle $j$, while the projectile moves in the gun, so that at the moment when the projectile leaves the muzzle of the gun, the axis of the bore, and therefore also the tangent to the trajectory at $G$, makes an angle $\phi' + j = \phi$ with the line of sight (or $\phi + \epsilon$ with the horizontal). The angle $j$ is always very small, but may be positive or negative. In aiming the gun, the (known) jump must be taken into account. In case gun ($G$) and target ($T$) be in the same horizontal plane, $T$ coincides with $N$, the "range" coincides with the "horizontal range," because $\epsilon = 0$, and the angle of departure is equal to the quadrant angle of departure, the angle of elevation equal to the quadrant angle of elevation. For the parabola in this case the angle of fall is equal to the angle of departure, while for the ballistic curve the angle of fall is then greater than the angle of departure.—The angle of site, $\epsilon$, is counted positive when target lies higher than gun, negative when lower.

## PROBLEMS

1. For the following quadrant angles of elevation, jump, site, find the angle of departure and the quadrant angle of departure. Draw curves showing all angles. (Alger, p. 26).

| Data | | | Answers | |
|---|---|---|---|---|
| $\phi' = 2°$ | $j = +\ 5'$ | $\epsilon = 15°$ | $\phi = 2°\ 5'$ | $\epsilon + \phi = 17°\ 5'$ |
| $3°$ | $-\ 3'$ | $12°\ 15'$ | $2°\ 57'$ | $15°\ 12'$ |
| $3°$ | $-\ 7'$ | $-\ 10°\ 30'$ | $2°\ 53'$ | $-\ 7°\ 37'$ |
| $4°$ | $+\ 6'$ | $-\ 9°\ 37'$ | $4°\ 6'$ | $-\ 5°\ 31'$ |
| $6°$ | $-\ 8'$ | $-\ 6°\ 22'$ | $5°\ 52'$ | $-\ 0°\ 30'$ |

2. An observation balloon is about 3000 ft. above the surface of the earth; its horizontal distance from an enemy gun is 4000 yds. Find the angle of site.

3. A gun is to fire over a hill 270 ft. high. The horizontal distance of the crest of the hill from the gun is 700 yds. How large must the angle of departure be, at least? Answer: $7°20'$.

4. A target is at a horizontal distance of 3700 yds. from the gun, and is 200 yards lower than the gun. Find the angle of site and the distance in a straight line from the gun to the target (range).

Since we do not assume any knowledge of analytic geometry on the part of the student, the use of coordinates must be briefly explained by the instructor if the following formulae and problems are taken up. The derivation of the formulae involves analytic geometry and some calculus so that they must be accepted without proof. This set is inserted because it affords good exercise in working with trigonometric formulae and because the artilleristic meaning of the problems is very clear.

In figure I let $x$, $y$ (measured in feet) be the coordinates of any point, $P$, of the trajectory in vacuo, $\psi = \phi + \epsilon$ the angle of departure, $t$ the time of flight (in seconds) until the projectile reaches $P$, $V$ the initial velocity (in feet per second), and $g = 32.2$, then the following formulae hold:*

$$x = t \cdot V \cdot \cos\psi \qquad\qquad y = t \cdot V \cdot \sin\psi - \tfrac{1}{2} g t^2.$$

Eliminating $t$, we obtain the relation between $x$ and $y$:

$$y = x \cdot \tan\psi - \frac{g \cdot x^2}{2V^2 . \cos^2\psi}.$$

From this the horizontal range $X$ (in feet) is obtained by assuming $y = 0$:

$$X = \frac{V^2 \cdot \sin 2\psi}{g}.$$

The total time of flight $T$ (for the horizontal range $X$) is given by

$$T = \frac{X}{V \cdot \cos\psi} = \sqrt{\frac{2X \cdot \tan\psi}{g}}.$$

## PROBLEMS

1. The data being as given in the first two columns of the following table, find the results, in vacuum, required by the other columns. (Alger, p. 32).

*Alger, p. 28.

| Initial Velocity $V$ (f. s.) | Angle of Departure $\phi + \epsilon = \psi$ | Horizontal Range $X$ (yds.) | Time of Flight $T$ (secs.) |
|---|---|---|---|
| 1000 | 5° 34′ | 1999 | 6.03 |
| 1100 | 4° 35′ | 1995 | 5.46 |
| 1250 | 3° 30′ | 1971 | 4.74 |
| 1400 | 2° 10′ | 1533 | 3.29 |
| 1500 | 7° 28′ | 6002 | 12.11 |
| 1750 | 8° 12′ | 8951 | 15.50 |
| 2000 | 12° 30′ | 17500 | 26.89 |
| 2400 | 7° 40′ | 15767 | 19.89 |
| 2600 | 3° 10′ | 7719 | 8.92 |
| 2900 | 16° 40′ | 47840 | 51.66 |

2. In the present war the Germans bombarded Paris from the Gobain Forest, about 70 miles from Paris. Show that, in vacuo, and assuming $g = 32.2$, the initial velocity must be at least between 3449 and 3450 f. s., and that the corresponding time of flight would be 151.5+ sec. (The expression for $X$ shows that for a given $V$ the range is greatest for $\psi = 45°$).

3. The data being as given in the first three columns of the following table, find the result, in vacuum, required by the fourth and fifth columns (see Fig. 1), (Alger, p. 33).

| Initial Velocity $V$ (f. s.) | Angle of Departure $\phi + \epsilon = \psi$ | $t$ (secs.) | $x$ (yds.) | $y$ (ft.) |
|---|---|---|---|---|
| 1000 | 5° 34′ | 3.01 | 999 | 146 |
| 1100 | 4° 35′ | 2.73 | 998 | 120 |
| 1250 | 3° 30′ | 2.37 | 986 | 90 |
| 1400 | 2° 10′ | 1.64 | 765 | 44 |
| 1500 | 7° 28′ | 6.05 | 2999 | 590 |
| 1750 | 8° 12′ | 5.00 | 2887 | 846 |
| 2000 | 12° 30′ | 20.00 | 13017 | 2218 |
| 2400 | 7° 40′ | 10.00 | 7929 | 1592 |
| 2600 | 3° 10′ | 8.00 | | |
| 2900 | 16° 40′ | 30.00 | | |

In the first five questions of this problem, and in the eighth, $y$ is practically the maximum ordinate.

4. A body is projected in vacuum with an angle of departure of 45°, and an initial velocity of 200 f. s. Compute the coordinates of its position after 6 seconds.

Ans.: $x = 848.5$ ft., $y = 268.9$ ft.

5. The measured range in air of a 12″ shell of 850 pounds weight, fired with 2800 f. s. initial velocity, and an angle of departure of 7° 32′, was 11,900 yds., and the time of flight was 19.5 seconds. What would the range and time of flight have been in vacuum? (Alger, p. 34).

Ans.: $X = 21097$ yds., $T = 22.8$ seconds.

6. Vigneulles, in the Saint Mihiel salient in France, is about 24 miles from the German fortress of Metz. Under what angle of departure would an American 12″ gun with initial velocity 2800 ft. per sec. have to be fired at *V*. to hit *M*. (neglecting the air resistance)?

Ans.: $\psi_1 = 15°40.9'$, $\psi_2 = 74°19.1'$. Explain why there are two answers. Would $\psi_1$ ($\psi_2$) have to be increased or decreased when the air resistance is taken into account?

Problems of the type given in this section will make clear to the student the mathematical background of problems dealing with "danger space" and "clearing the crest" or "firing over a mask." However, such problems are treated in Field Artillery by very simple methods of approximation and are for this reason omitted here.

# SECTION B.

## *Definition and Simple Applications of the "mil"—The Parallax.*

A first difficulty which the student will encounter in artillery work lies in the fact that the U. S. Field Artillery measures angles generally in so-called "mils".* The sighting instruments of the guns are graduated in this unit, instead of degrees, and the tables are all made out accordingly. The mil will therefore have to be carefully considered in a trigonometry course which is to prepare for artillery service. In the Heavy and Coast Artillery the conventional system of measuring angles in degrees, minutes and seconds is used together with the mil system. It should also be noted that the Field Artillery, which has up to the present measured lengths in yards, is, as far as length measurements are concerned, in a stage of transition, since, in order to agree with French practice, lengths are in the future to be measured in meters.

According to some text-books the mil was originally defined as one one-thousandth of a radian.**

There would thus be $2000.\pi = 6283$ (approx.) mils in $360^\circ$. This would be a very inconvenient unit for numerical computations. The mil actually adopted in the army is the sixteen-hundredth part of a right angle:

$$1 \text{ mil} = \frac{1}{1600} \text{ right angle} = .05625^\circ = \frac{27'}{8},$$

6400 mils $= 360^\circ$, 3200 mils $= 180^\circ$, 1600 mils $= 90^\circ$.

The student may verify that the mil is about 4 seconds (that is, about 2 per cent), smaller than 1-1000 radian.

The following is quoted from official instructions of the United States Army:

*Definition*: *All U. S. mobile artillery sights will be graduated clockwise in mils. A mil is 1-6400 of a circle. The arc which subtends a mil at the center of a circle is, for practical purposes, equal to 1-1000 of the radius. The arc and its tangent are nearly equal for angles not greater than 350 mils.****

---

*Compare for this section: *Bishop*, p. 47 ff.; *Moretti-Danford*, pp. 57, 58 for definitions of mil and parallax; numerous applications pp. 62-130; Gunnery, pp. 33, 34, 38; Manual pp.115-121.

**According to other text books the mil was first defined as arc tan .001. The difference between this angle and the angle 1-1000 radian is only about one millionth of one minute.

***350 mils $= \frac{350 \cdot 360^\circ}{6400} = 19\frac{11^\circ}{16} = 20^\circ$—.

1. Change 1 mil to degrees; to minutes; to seconds.
2. Change to mils:
$1^\circ$; $1'$; $1''$; $60^\circ$; $200^\circ$; $75^\circ\ 20'$; $142^\circ\ 35'$; $40'$; $5^\circ\ 10'$; $17'$; $1^\circ\ 25'$.
3. Change to degrees and draw the angles:
100 mils; 80 mils; 2400 mils; 1360 mils; 5200 mils; 50 mils.
4. Given a triangle with angles $120^\circ$; $51^\circ\ 30'$; $8^\circ\ 30'$, change all angles to mils and check by $180^\circ = 3200$ mils.
5. Given a triangle with angles 1280 mils; 760 mils; 1160 mils. Change all angles to degrees and check.

In military textbooks an abbreviation for "mil" does not seem to be in use. Frequently the angle in mils is given without any notation, as $A = 310$.

The last two sentences quoted in the official instructions point toward the most important applications of the mil, which we now discuss.

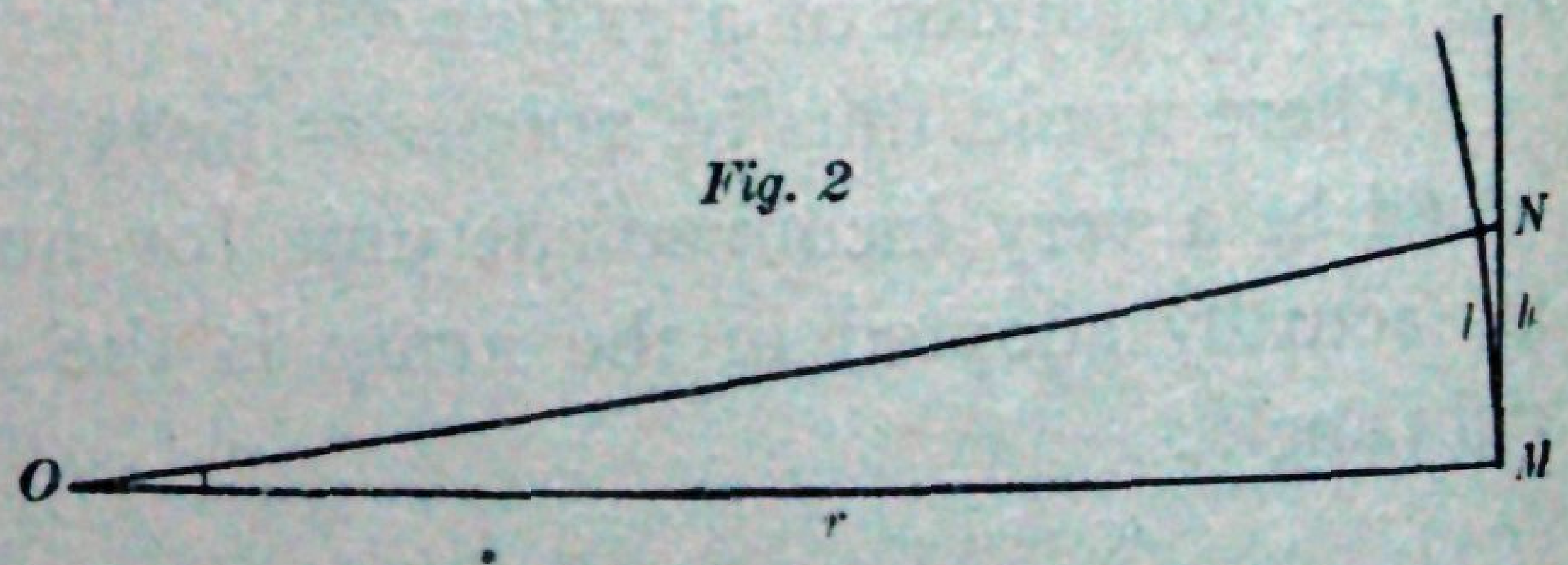


*Fig. 2*

In the right triangle $OMN$ (Fig. 2) let $\angle O = a$ radians $= k$ mils, $OM = r$, $MN = h$, and let $l$ be the arc between $OM$ and $ON$ of a circle about $O$ as center. Then $\tan O = h:r$. For very small angles $O$ the ratio $h:l$ is very nearly unity; with increasing angles $O$ the ratio $h:l$ also increases; but for $O = 20^\circ$ the fraction $h:l$ has only reached the value 1.04+. For $O = 15^\circ$, $h:l = 1.02+$; for $O = 10^\circ$, $h:l = 1.01+$; for $O = 5^\circ$, $h:l = 1.003-$. The error made by replacing the arc by the tangent is thus about four to five per cent for a 350 mil angle, about two per cent for a 270 mil angle, about one per cent for a 150 to 200 mil angle, about ⅓ per cent for a 75 to 100 mil angle. Therefore $h:r = l:r$ (approx.) for small angles $O$.* But $l:r = a$, and $a = k:1000$ (appr.), hence, for small angles, $h:r = k:1000$ (appr.),

$$k = h:(r/1000).$$

In the most important applications of the mil in gunnery $r$ is the gun range and measures usually some thousands of yards, while $h$ is comparatively small (height of a tree, or of a hill, or a high building,

*Trigonometrically, $h:r = l:r$ (appr.) expresses that $\lim (\tan x : x) = 1$ when $x$ approaches 0 and is measured in radians.

etc.; or it may be a comparatively short line in the horizontal plane, such as the distance from the gun to the battery-commander's station).

We have thus the important

RULE: *If r and h are both measured in the same unit, then the angle subtended by h in Fig. 2 is* $h:\left(\frac{r}{1000}\right)$ *mils (appr.).*

According to this rule 1 yd. subtends at 1000 yds. an angle of one mil, 2 yds. subtend at 2000 yds. an angle of 1 mil, etc.

A sighting instrument graduated in mils enables an observer to determine immediately each of the quantities $r$, $h$, $k$ from the other two. In estimating the error caused by applying the rule stated above, two sources of error must be considered. Firstly, the mil is used as if it were exactly 1/1000 radian, thus causing a constant error of about two per cent. Secondly, we have an error which varies with the angle and which is caused by replacing $l:r$ by $h:r$. These two errors tend to counteract each other. Therefore the rule gives correct results when the error from the second cause is as large as the error from the first cause, that is, about two per cent. This happens for an angle in the neighborhood of fifteen degrees (about 270 mils), as we know. For this question compare *Whittemore.*

## PROBLEMS *

1. Given that a target is 3000 yds. distant from the gun and 200 feet higher than the gun. Find the angle of site. (See Fig. 1).

2. Find the angle of site when
   (a) range = 2500 yds., target 200 yds. higher than gun.
   (b) range = 4700 yds., target 150 ft. lower than gun.

   Ans. (b): — 10.6 mils.

3. A tower of 150 ft. height subtends at the gun an angle of 30 mils. Find the distance from gun to tower.

   Ans.: About 5000 feet.

4. A tree subtends at a distance of 1500 ft. an angle of 60 mils. How high is the tree?

5. Find the error made in finding $k$ in the following problems by the rule given in the text.

*Most problems involving the mil are conveniently worked by slide-rule.

(a) $r = 2000$ yds, $h = 100$ ft.
(b) $r = 2000$ yds, $h = 400$ ft.
(c) $r = 2000$ yds, $h = 1000$ ft.
(d) $r = 2000$ yds, $h = 2000$ ft.
(e) $r = 2000$ yds, $h = 6000$ ft.
(f) $r = 2000$ yds, $h = 10000$ ft.
(g) $r = 2000$ yds, $h = 20000$ ft.

## PARALLAX. CORRECTION OF PARALLAX FOR OBLIQUITY

*Definition: The parallax of a line at a point is the angle subtended by the line at the point.*

In Field Artillery this angle is measured in mils.

Assume first that the point $O$ at which a line $MN = h$ subtends an angle of $k$ mils lies on the perpendicular bisector of $h$. (See Fig. 3).

To find the distance $OP = l$ of $h$ from $O$, we should have

$$l = h/2 \cdot \cot k/2.$$

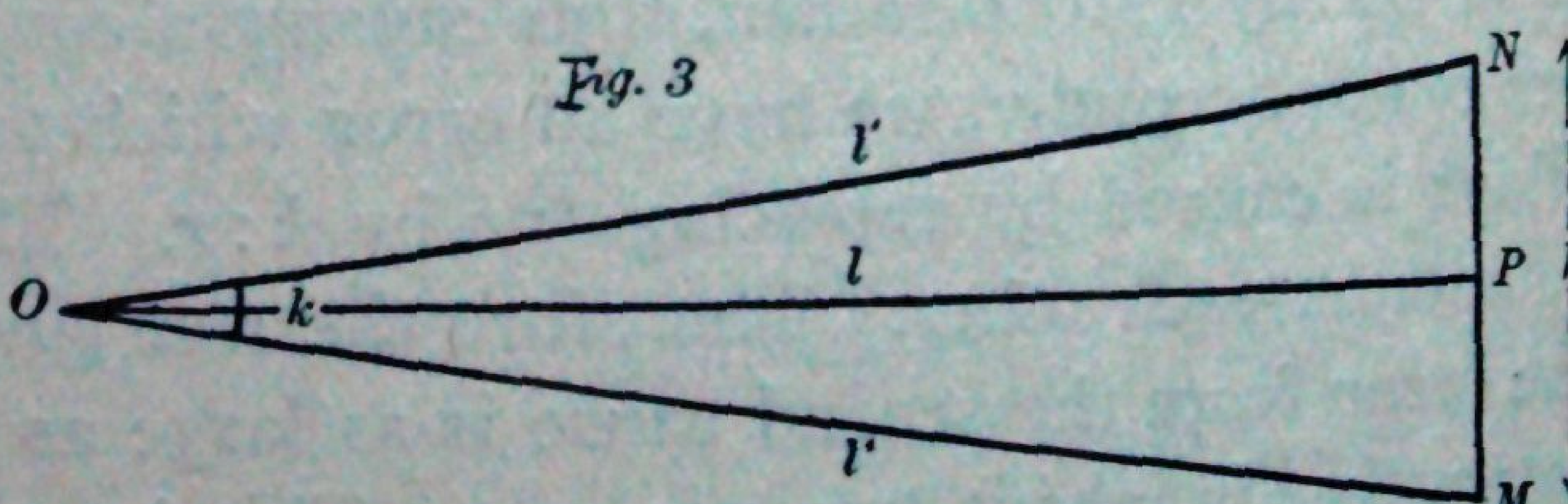


Fig. 3

However, when $l:h$ is a small fraction, we may apply the rule of page 13 and obtain:

$$k/2 = (h/2) : (l/1000), \text{ or } l:1000 = h:k, \text{ or}$$

$$l = \frac{1000\,h}{k} = h : \left(\frac{k}{1000}\right)$$

Since for $h:l$ small, $l:l'$ nearly unity, we may in this case also use:

$$l' = \frac{1000\,h}{k} \text{ (appr.).}$$

*We assume from now on the fraction $h/l$ so small that our approximation formulae hold.*

If $O$ does not lie exactly on the perpendicular bisector of $h$, but so close to it that $\triangle ONM$ is approximately isosceles, $(1000\,h) : k$ will still give a good approximation for the distance of $O$ from $h$.

If $\triangle ONM$ is not approximately isosceles, this expression cannot be used. We proceed then as follows:

Consider (Fig. 4) the parallax of $MN = h$ at $O$. Draw the perpendicular bisector $PO'$ of $MN$, making $PO' = PO = l$ (say).

Then $\angle NO'M = h : \frac{l}{1000}$ mils (appr.), while $\angle NOM$, the angle we are interested in, is obviously smaller. Therefore, a correction must be applied to $O'$ to obtain $O$, (or to $O$ to obtain $O'$). For this purpose the angle $MPO = \gamma$, the so-called *"angle of obliquity"*, is introduced.

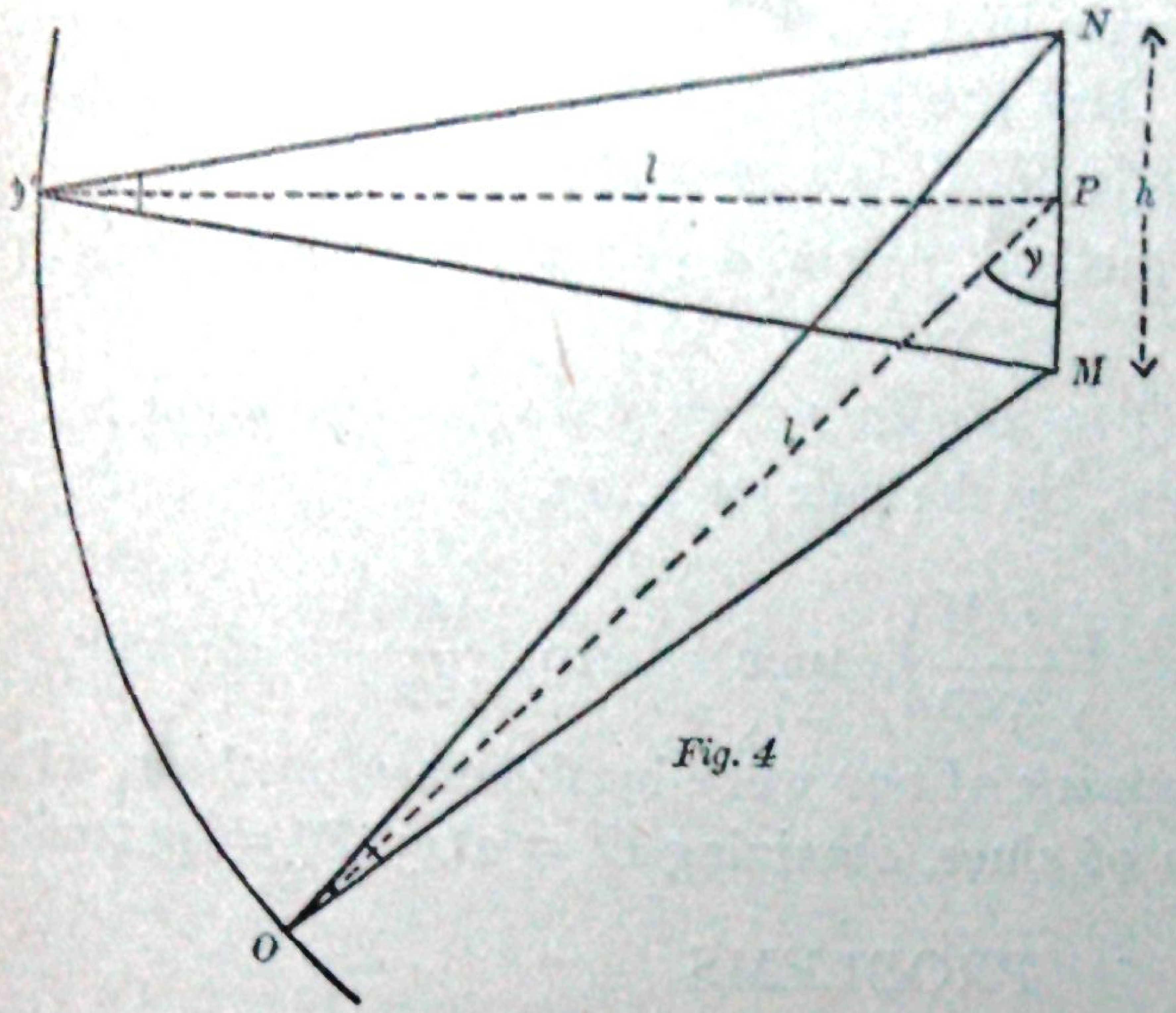



*Fig. 4*

It is easily seen that if we assume a relation of the form $O = f(\gamma) \cdot O'$, then $f(\gamma)$ increases from 0 to 1 when $\gamma$ increases from 0° to 90°; the general behaviour of the factor $f(\gamma)$ is therefore similar to the sine function.

In Field Artillery, the following values are usually chosen, with corresponding rough interpolations:

| $\gamma$ | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| $f(\gamma)$ | 0 | .5 | .7 | .9 | 1 |

The angle $\gamma$ is frequently estimated; *its accurate value is not required in general.**

The factor $f(\gamma)$ is the *"factor of obliquity"*; its application gives the *"correction for obliquity."* When greater accuracy is required, small "obliquity tables" are used.

*Example*: Given $h = 300$ yds., $OM = 4400$ yds., $\angle MNO = 45°$. To find $\angle MON$.

*Solution*: First method. Let $\angle MON = k$ mils.

Since $MN$ is small as compared with $OM$, *the angle of obliquity $\gamma$ will be approximately equal to $\angle MNO$*; we assume $\gamma = 45°$.

In $\triangle O'MN$, if $k$ denotes the number of mils in $O'$,

$$k' = 300 : \frac{4400}{1000} = 68.2 - \text{mils.}$$

*It may therefore be replaced by angle $ONM$, if convenient (since $h:l$ is assumed small).

But $k = k' \sin \gamma = 68.2 \cdot .7 = 47.7 = 48 -$ mils.

Another method for treating the correction for obliquity is often employed, for example in the problem of determining the "deflection" in indirect firing (see p. 27). This method will be sufficiently explained if we apply it to the example just worked out. ($M$ may be assumed to be a gun, $O$ the target, $MO$ the range, $N$ the "battery commander's station"; the required angle $MON$ is a so-called "offset").

*Solution*: Second method. Drop a perpendicular $MN_1$ from $M$ on to $NO$, then

$MN_1 = h \cdot \sin MNO = 300 \cdot \sin 45^\circ = 300 \cdot .7 = 210$ (appr.).

From $\triangle\ OMN_1$, then, by the rule of p. 13.

$$k = h : \left(\frac{ON_1}{1000}\right) = h : \left(\frac{OM}{1000}\right) \text{(appr)} = 210 : \frac{4400}{1000} = 48 - \text{mils.}$$

To estimate the accuracy of our work by these two methods, solve $\triangle OMN$ by the theorem of sines, obtaining $O = 2^\circ 45' 48'' = 49.1$ mils.

## PROBLEMS

1. A line of length $h$ yards has a parallax of $k$ mils at a distance $l$ yards from the line. The angle of obliquity is $90^\circ$. From any two values in each line of the following table find the remaining one by the rule of p. 13.

| $h$ | $l$ | $k$ |
|---|---|---|
| 100 | 2000 | 50 |
| 200 | 1000 | 200 |
| 330 | 2200 | 150 |
| 220 | 2500 | 88 |
| 270 | 500 | 540 |

2. Find in the preceding problem for each question the error in $k$ due to the use of the method of approximation.

3. The quantities $h$, $l$, $\gamma$, $O = k$ mils, have the meaning indicated in Fig. 4. Solve in each line for the unknown quantity:

| $h$ | $l$ | $\gamma$ | $k$ |
|---|---|---|---|
| 200 | 1500 | 60° | ? |
| 400 | ? | 45° | 60 |
| ? | 900 | 30° | 100 |

4. A ship of 650 ft. length is sailing due northwest. For an observer on another boat due west her parallax is 55 mils. How far are the ships apart?

Ans.: 8400— ft. (taking the corection factor .71).

5. A bridge crosses a river 850 ft. wide; the river flows in a straight course. From a boat on the river the bridge appears under an angle of 4½ degrees. Find approximately the distance from the boat to the bridge.

Ans.: $\frac{850}{80} \cdot 1000$ ft.

## SECTION C.

## CALCULATION OF FIRING DATA FOR DIRECT AND INDIRECT FIRE.

*For greater simplicity we assume in this section throughout that the gun (more accurately, the muzzle of the gun), the target, and, as far as they will be used, the "battery commander's station" and the "aiming point" all lie in a horizontal plane.*

Moderate differences in altitude between the gun and the target do not offer serious difficulties in practice.

In pointing a gun it is first necessary to know the range and direction of firing.* The determination of these quantities is the only problem which we shall discuss in this section. When range and direction are known, "range tables", constructed for each type of gun, give the angle of elevation under which the gun must be fired.

In the Field Artillery the range is determined by rough computations or measurements, and an error of a hundred yards or more is apparently accepted as normal; corrections are based on actual observations of the results of firing. In the Coast and Heavy Artillery, on the other hand, every effort is made to secure a hit with one of the first shots.

We abstract entirely from relatively small, but very important corrections which must be made in pointing the gun and which are due to rifling, wind, etc. for the *direction,* and to wind, temperature, air-pressure, etc. for the *range.*

### OUTLINE

I. Determination of range when target is visible from gun.

II. Determination of range when target is not visible from gun.

III. Determination of direction of firing when target is visible from gun.

IV. Determination of direction of firing when target is not visible from gun. (Deflection).

### DETERMINATION OF RANGE.

For the determination of the range several methods are available of which we mention the following:

I. *Target visible from gun; find range.*

Ia. *From the maps.*

---

*Compare for this section: *Bishop*, pp. 49-59; *Moretti* and *Danford*, pp. 68-130; *Spaulding*, Ch. V; Gunnery, Ch. V; Manual, pp. 115-128.

For the whole Western Front in Europe there exist extremely accurate maps of each "sector", covering the whole possible field of operations. Such maps are covered with a system of "index lines", that is, by two sets of parallel straight lines which divide the map into squares. The most detailed maps are on a scale of 1 : 5000, so that one square mile in nature is represented by about one square foot on the map. When the target is visible, its position on the map can be fairly accurately determined, and since the position of the gun on the map is likewise known,* the range is found either by actual measurement on the map or by using the Theorem of Pythagoras in an obvious way.

Ib. *By using range finding instruments.*

Theoretically, the simplest range finder is an instrument consisting of two telescopes joined by a rigid (horizontal or vertical) bar of known length.**

The telescopes are both focussed on the target and the angles read off which the lines of vision make with the horizontal (or vertical) bar. In the triangle formed by the bar and the two lines of vision one side and two angles are known and the required distance (one of the remaining sides) may be easily determined. (Since the range is large as compared with the distance of the telescopes, the parallax method with correction for obliquity would apply). However, this type of range finder is not sufficiently accurate, since a very small error in the angles causes a large error in the distance, on account of the short base. (Compare Ic). A type of optical range finder, based on the refraction of light in a system of prisms, is actually used.

Ic. *Trigonometric Methods and Use of Parallaxes.*

A point $C$ is selected (which we assume, for simplicity, to lie in a horizontal plane with gun and target) from which both gun $G$ and target $T$ are visible. The distance from gun to $C$ is measured, and the angles at $G$ and at $C$ in $\triangle GCT$ are observed. Then the range $GT$ is determined by the theorem of sines. Obviously this is again the method of Ib except that the base is now chosen arbitrarily.

The work is considerably simplified if $G$ is made equal to 90°, as is frequently possible. The problem then reduces to the solution of a right triangle.

---

*In Heavy Artillery, trigonometric (surveying) methods are frequently employed, when the position of the gun must be very accurately determined (with reference to fixed points on the map). We assume the location of the gun on the map to be known with sufficient accuracy.

**In one instrument, Berdan's range finder, a horizontal bar of six feet length is employed.

## PROBLEMS

1. Assuming $G = 90°$, find the range $GT$ for
   (a) $GC = 820$ yds., $C = 75°55'$.
   (b) $GC = 200$ yds., $C = 87°\ 5'$.
   (c) $GC = 100$ yds., $C = 88°25'$.

2. In problem 1, find in (a), (b), (c) the change in the range due to an increase in $C$ of $5'$.

3. $GC = 1500$ yds., $C = 72°30'$, $G = 86°20'$. Using the theorem of sines, find range $GT$.

4. $GC = 550$ yds., $C = 45°50'$, $G = 125°25'$. Find range $GT$.

*When GC is small as compared with the range, and G = 90°, the range may be found by the rule of p. 13.*

*Example*: From $G$ a line $GC$ of 150 yds. length is measured off at right angles to $GT$. The angle at $C$ is measured, $C = 1560$ mils. Find range $GT$.

*Solution*: $T + C = 90° = 1600$ mils, $T = 40$ mils. Applying our rule, we have, for range $= x$ yds.

$$150 : \left(\frac{x}{1000}\right) = 40,\ x = \frac{150 \cdot 1000}{40} = 3750 \text{ yds.}$$

(The true value of $x$ is $150 \cdot \cot 2°15' = 3818-$ yds.).

Compare for this kind of work the problems of pp. 13-14.

*When GC is small as compared with the range, and G different from 90°, the range may be found by using parallaxes and sufficiently accurate obliquity-factors.*

*Example*: From the gun $G$ a line $GC$ of 200 yds. length is measured off. $\angle G$ is found to be $43°52\frac{1}{2}' = 780$ mils, $\angle C$ is $133°52\frac{1}{2}' = 2380$ mils. Find range $GT = l$
(a) by solving the triangle $GCT$,
(b) by using the parallax method.

*Solution*: (a). From $T = 180° - (G+C) = 2°15'$,

$$l : \sin 133°52\tfrac{1}{2}' = 200 : \sin 2°15',$$

we find $l = 3672$ yds.

(b). Compare p. 15, first method. Fig. 5 is only schematic. The student is advised to draw a figure to scale. We assume that $GT = l$

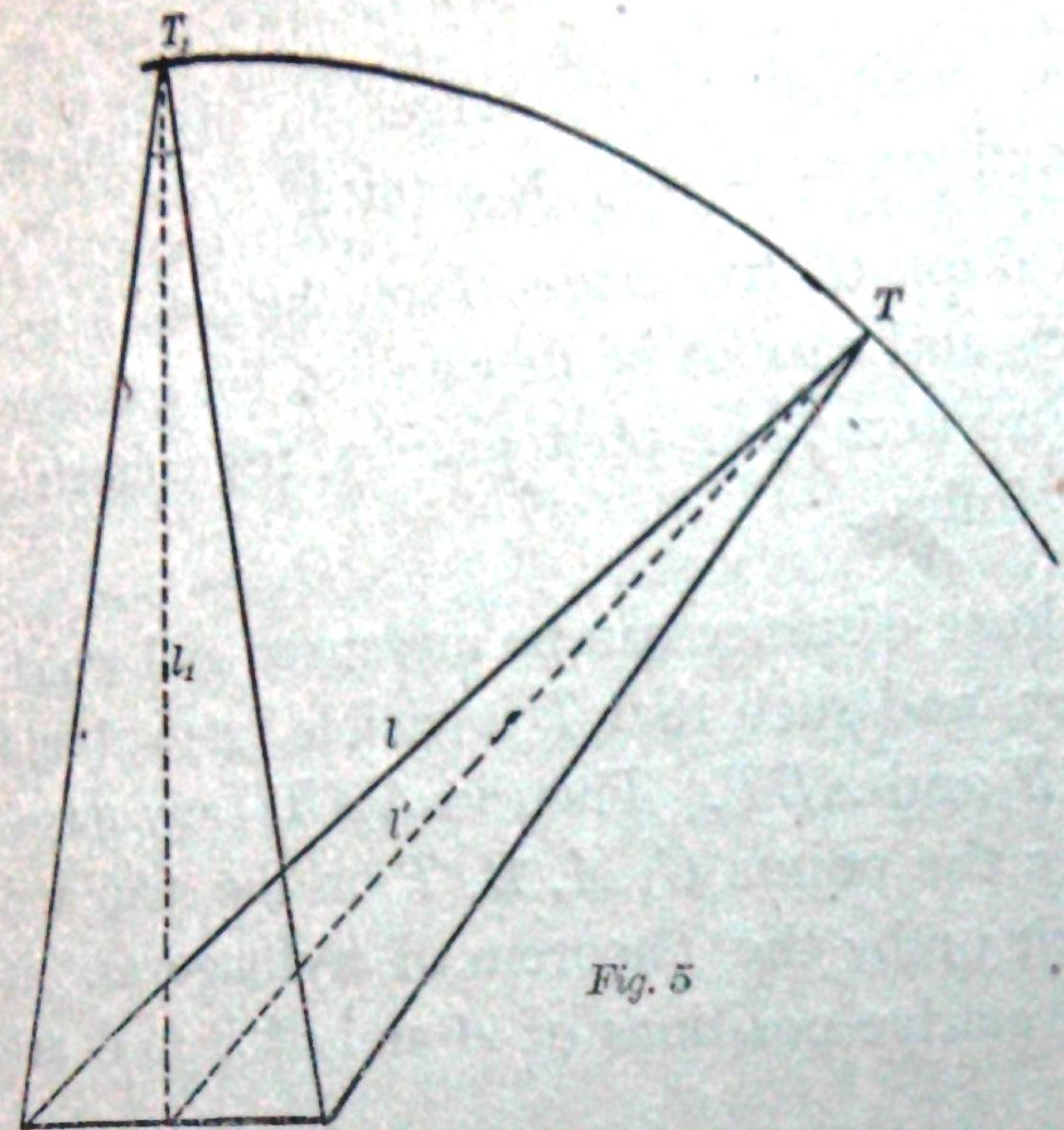



*Fig. 5*

may be replaced with sufficient accuracy by $DT = l'$ (in case the error thus committed is too serious, an estimated correction is easily applied). Since $GC$ is small as compared with $GT$, *the angle of obliquity* $TDC = \gamma$ *may be with sufficient accuracy taken to be equal to* $\angle TGC$. Then, by Section B, $\angle GT_1C = \angle GTC : \sin TGC$ (approx.). Taking $\angle TGC$ as $45^\circ$ (instead of $43^\circ 52\frac{1}{2}'$; in corrections for obliquity rough approximations to the angles are always considered sufficient), we have $T = 3200 - G - C$, and $T_1 = T : .71 = 40 : .71$ mils $= 56 +$ mils.

Therefore $56 = 200 : \dfrac{GT_1}{1000}$

$$GT_1 = GT \text{ (appr.)} = 3570 + \text{ yds.} = 3600 - \text{ yds.}$$

The error committed is about 100 yds.

## PROBLEMS

1. In the triangle formed by the gun ($G$), the target ($T$), and the point $C$ assume

(a) $GC = 300$ yds., $G = 90^\circ$, $C = 1520$ mils,
(b) $GC = 180$ yds., $G = 90^\circ$, $C = 1560$ mils,
(c) $GC = 600$ yds., $G = 90^\circ$, $C = 1400$ mils.

Find range $GT$ by solving the right triangle and also by using the rule of p. 13, and find the error.

2. In the triangle $GCT$ assume

$GC = 300$ yds., $G = 2000$ mils, $C = 1140$ mils.

Find the range $GT$ by using parallaxes, and determine the error committed.

Ans.: By the theorem of sines 4585 yds.; by parallax method 4500 (choosing obliquity factor .9).

II. *Target not visible from gun; find range.*

IIa. Frequently the position of the target on the map is known and the method indicated under Ia may be applied.

IIb. In case the position of the target $T$ on the map is not known with sufficient accuracy and cannot be determined for example by aeroplane observations or by aeroplane photography, trigonometric methods may be applied as follows:

Select two points $A$, $B$ whose distance can be measured and which are visible each from the other, and such that $T$ is visible from $A$ and from $B$. Measure angles $TAB$ and $ABT$. Then in $\Delta ABT$ any quantity can be determined. Enter the points $G$, $A$, $B$, $T$ on the map and find range $GT$ by measurement or by the Theorem of Pythagoras. It is of course assumed that the relative positions of $A$ and of $B$ to $G$ are known.

Other trigonometric methods are easily devised.

## DETERMINATION OF DIRECTION OF FIRING.

III. *Target visible from gun.*

Usually the gun and target are not visible one from the other. When the target is visible from the gun, it is possible to sight directly, taking afterwards in aiming the gun the necessary corrections into account. This is called "direct firing" or "direct laying."* In this case only the range has to be determined which may be done by one of the methods explained. Another method consists in entering gun and target on the map and determining the direction of firing by means of map and compass. (See *Moretti* and *Danford,* p. 113).

IV. *Target not visible from gun.—Aiming Point; Deflection.*

In this problem we may assume not only the range known but also the length of any other segment which may be useful, provided at least one end point of the segment can be reached by an observer. (By the methods explained in I and II).

IVa. Assume a point $B$ chosen as the "Battery Commander's Station", from which both gun and target are visible and such that $BG = c$ can be measured. Angle $GBT$ is measured at $B$. In $\Delta BGT$ two sides (range and $c$) and the angle opposite one side are known

*See Remarks, p. 29.

so that $\angle BGT$ can be determined by the theorem of sines. The gun first aims in the direction $GB$, and then swings through the angle $BGT$. Since in practice the range is usually some thousands of yards while $c$ is a few hundred yards, the given angle lies opposite the larger side and there is no ambiguity.

It is clear that (for $c/range$ sufficiently small) the parallax method may be used to determine $T$ and hence $G = 3200 - B - T$ mils. If, in particular, the angle at $B$ is not far different from $90°$, no correction will be required for obliquity.—IVa. is not usually applied in practice, in spite of its theoretical simplicity (compare Remarks, p. 29). It may serve to arouse interest in the solution of triangles.

## PROBLEMS

1. Review problem 5, page 13.
2. For each set of data in the following table find $\angle BGT$ by trigonometry and also by the method of parallaxes. In each case find the error caused by the method of parallaxes. In which cases does inspection show that the method of parallaxes will not give satisfactory results? (See "Definition of mil," p. 11; compare pp. 14, 15 and examples pp. 15, 16, 20).

| | $BG$ | $GT$ | $GBT$ |
|---|---|---|---|
| (a) | 250 yds. | 3200 yds. | 60° |
| (b) | 320 ft. | 1050 yds. | 45° |
| (c) | 700 ft. | 900 yds. | 45° |
| (d) | 275 yds. | 300 yds. | 30° |
| (e) | 400 ft. | 1800 yds. | 60° |
| (f) | 1250 yds. | 750 yds. | 30° |

### IVb. *Use of the "Aiming Point".*

Assume again a point $B$ (Battery Commander) from which $G$ and $T$ are both visible and such that $BG$ is easily measured. ($BG$ will usually be chosen not more than a few hundred yards in length).

Next, a point $P$ is selected, the so-called *Aiming Point*, which must be clearly visible from $G$ and from $B$. ($P$ is usually chosen as distant as possible from $B$ and $G$ consistent with visibility; $GP$ and $BP$ will therefore measure up to several thousand yards).

Since all lengths which we shall use may be assumed known, our problem will consist in determining certain angles. The idea is to have the gun first aimed in the direction $GP$, that is, as if the (visible) aiming point were the target, and then to swing the gun through the angle $PGT$, where the angle $PGT = d$ is the quantity which is to be determined from the known data. (See Fig. 6).

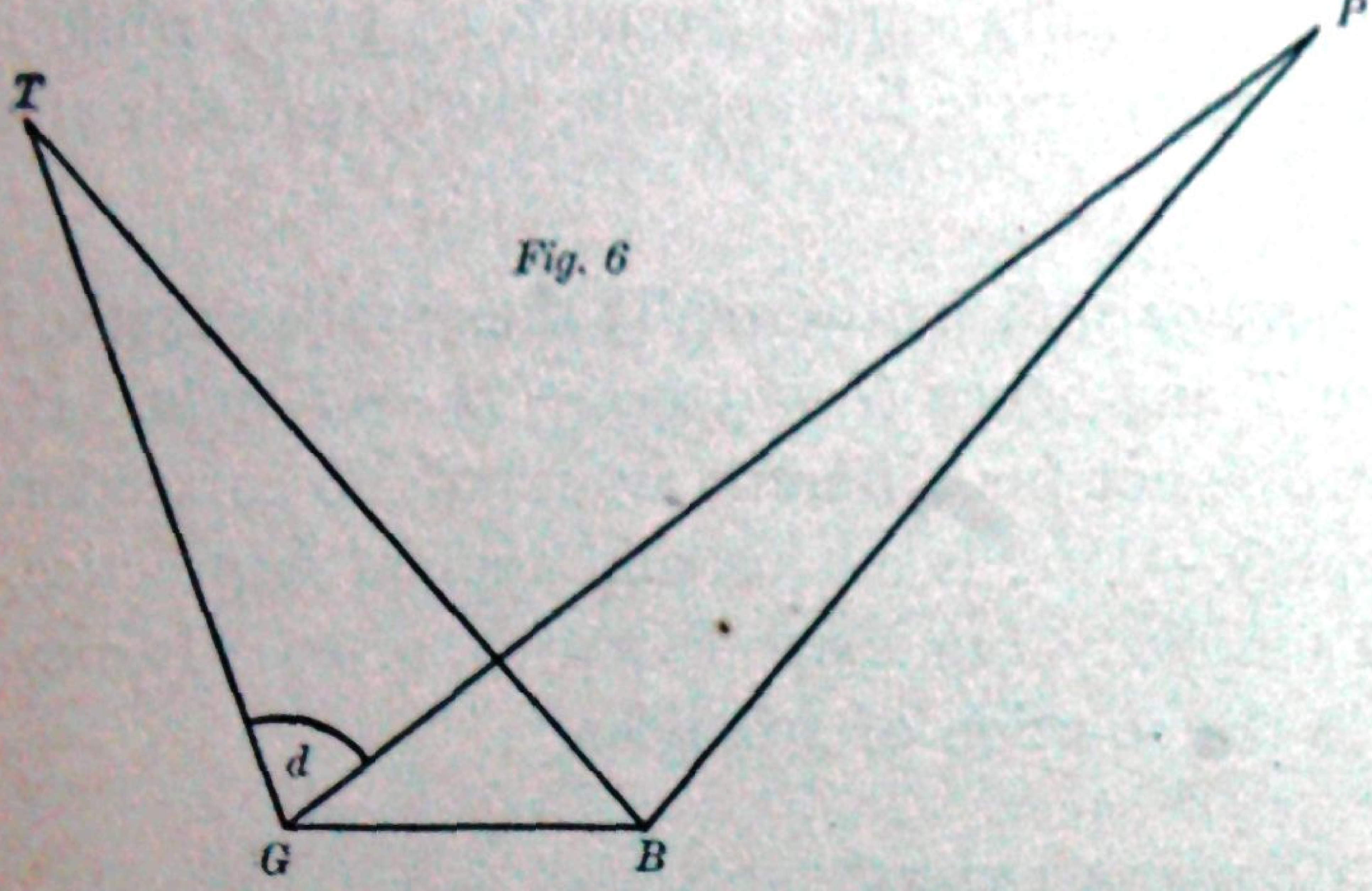



Fig. 6

*Definition*: *The angle $PGT = d$ is called the angle of deflection.* It is announced in mils, at least in the Field Artillery.

The deflection is a fundamental quantity in artillery work, and its determination one of the most important mathematical problems in Field Artillery Service.

In Field Artillery, *the deflection is the angle $PGT$ measured counter clockwise.* The angle $PBT = A$ is likewise measured counter clockwise. Both angles are measured from 0 mils to 6400 mils. The (small) angles $GTB = T$ and $GPB = P$ are the "offset angles" or the "offsets," and are counted positive.

The figures of p. 25 will illustrate the manner of measuring the angles.

In each case the equation printed with the figure is read off without any difficulty. For example, in 7a, $A + P = d + T$; in 7c, $(360° - A) + d + T + P = 360°$; in 7f, $(360° - d) + T = (360° - A) + P$.

*Rule*: *One obtains, ~~for all positions of G and T,~~ however B and P are chosen a relation of the type*

$$d = A \pm P \pm T,$$

and each of the four possible combinations of signs actually can occur, as our figures show. Many rules exist to decide quickly which combination must be chosen in a given case. In a trigonometry course it is sufficient to derive the relation from the figure in any particular problem.

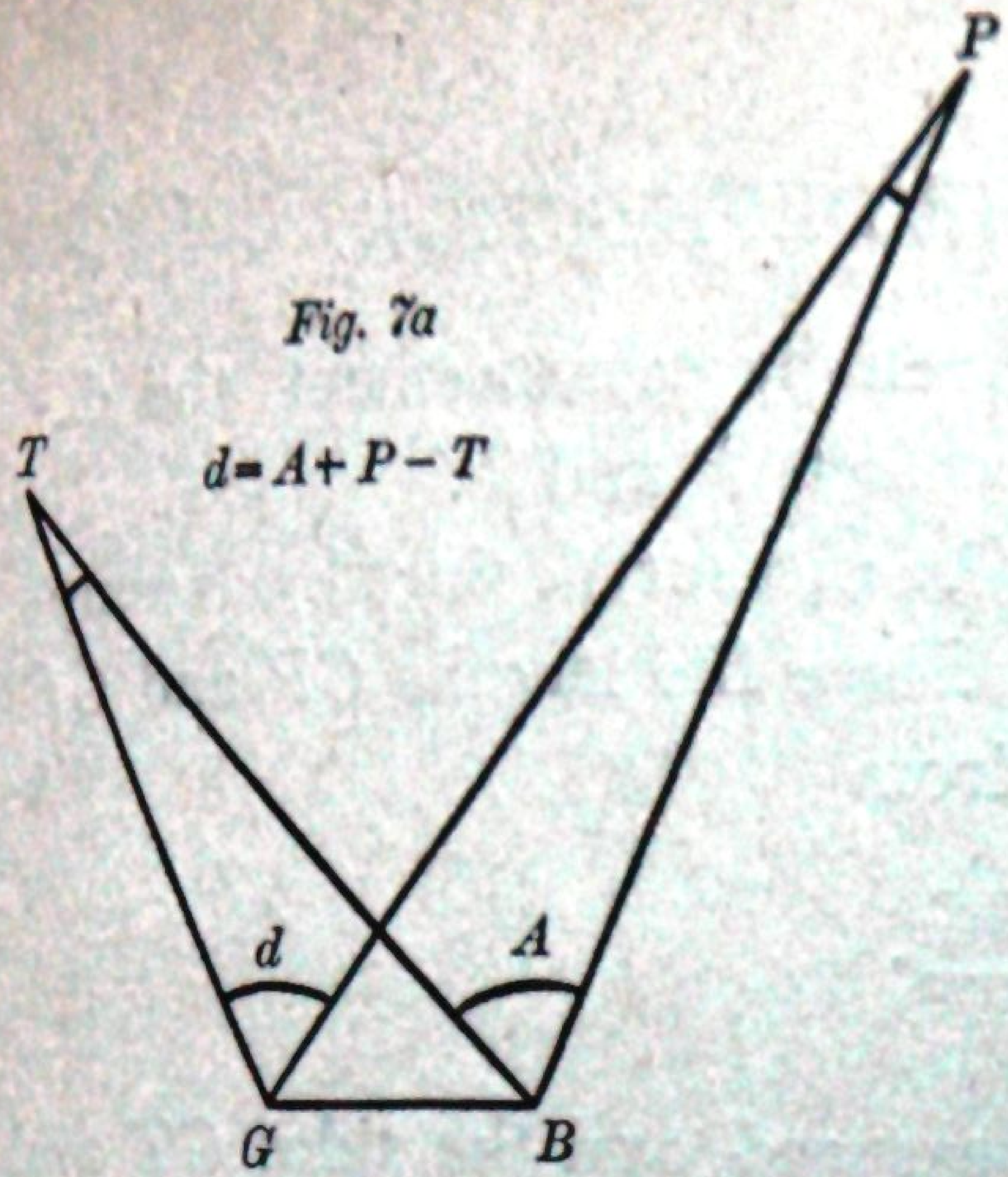



Fig. 7a

$d = A + P - T$

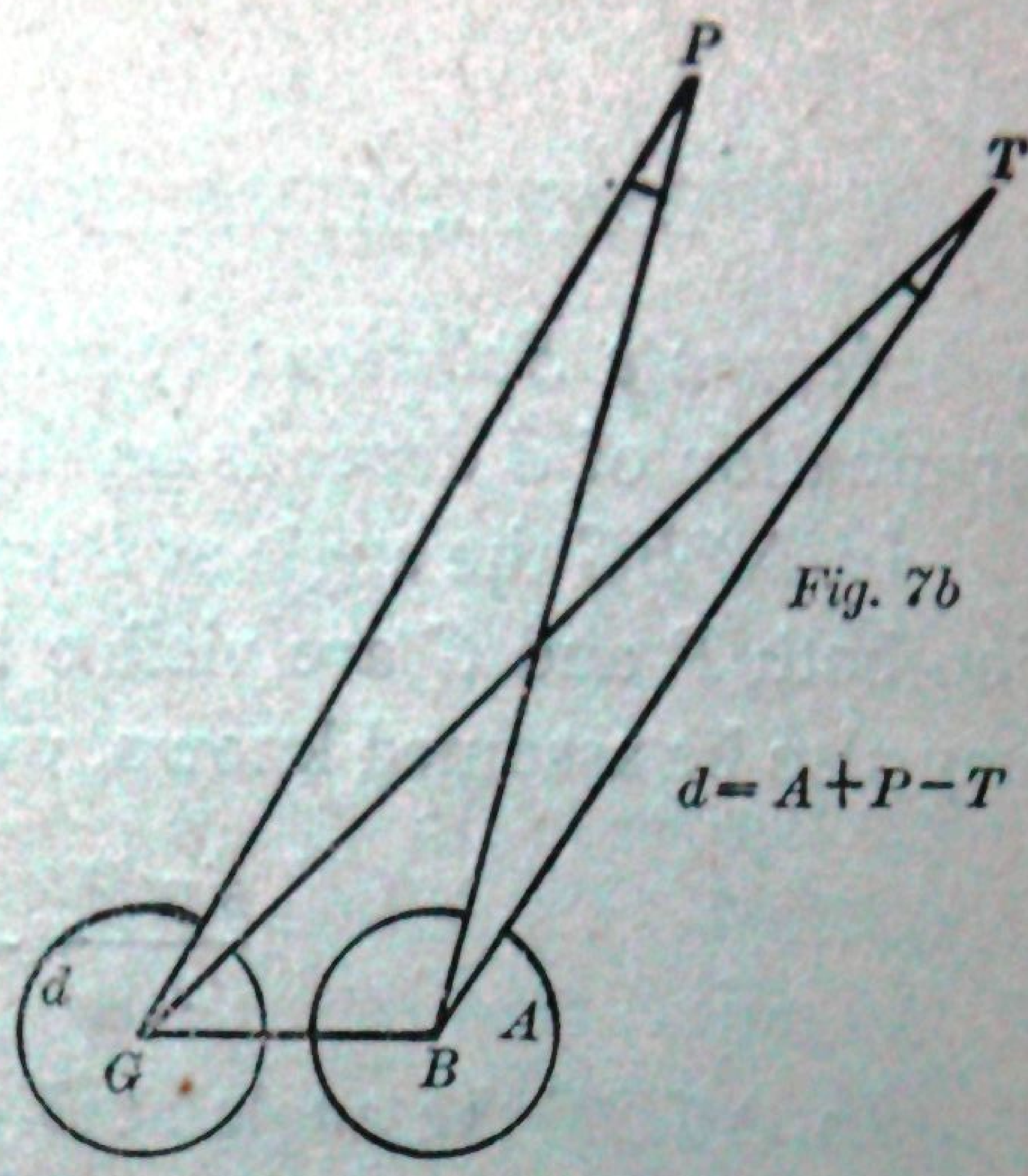



Fig. 7b

$d = A + P - T$

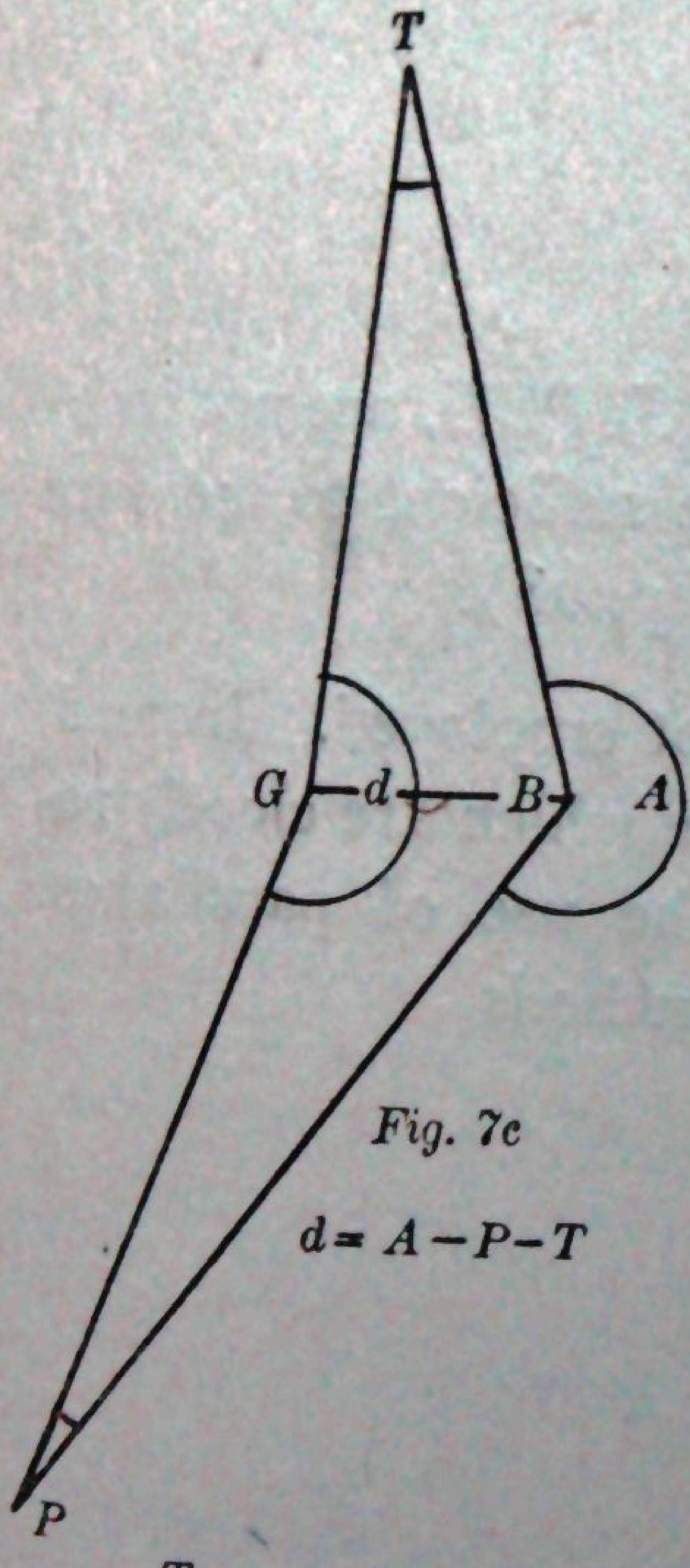



Fig. 7c

$d = A - P - T$

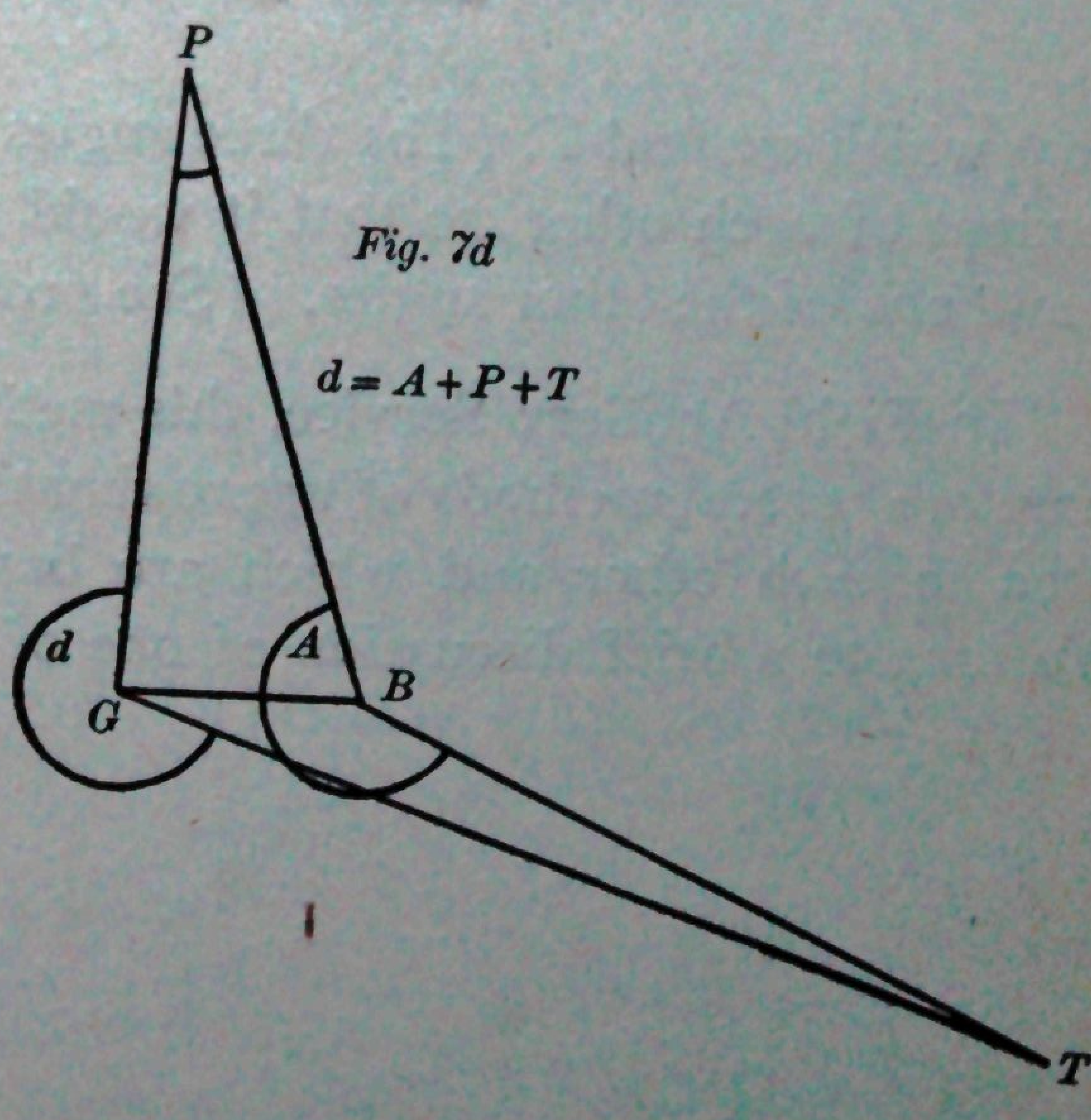



Fig. 7d

$d = A + P + T$

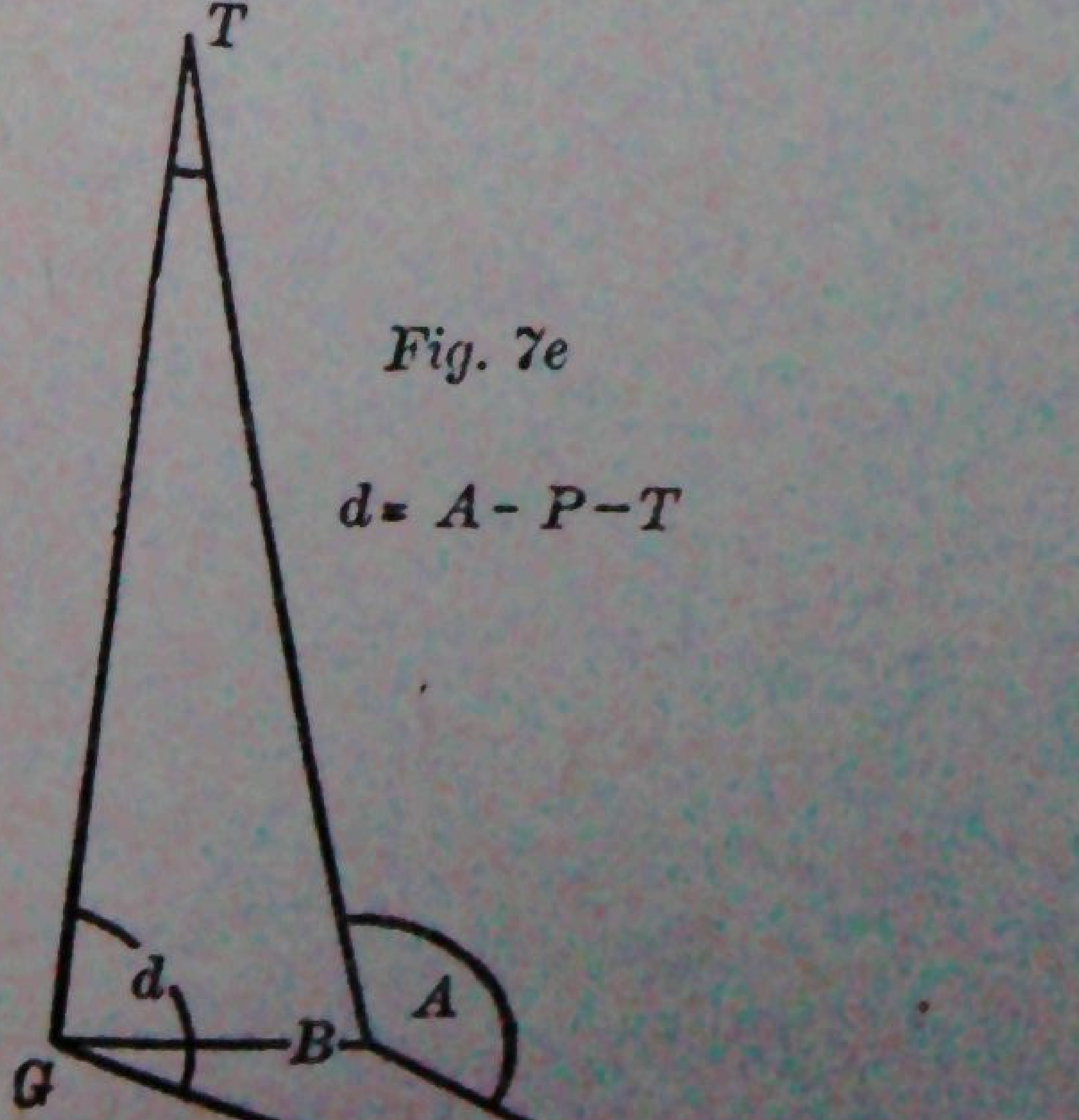



Fig. 7e

$d = A - P - T$

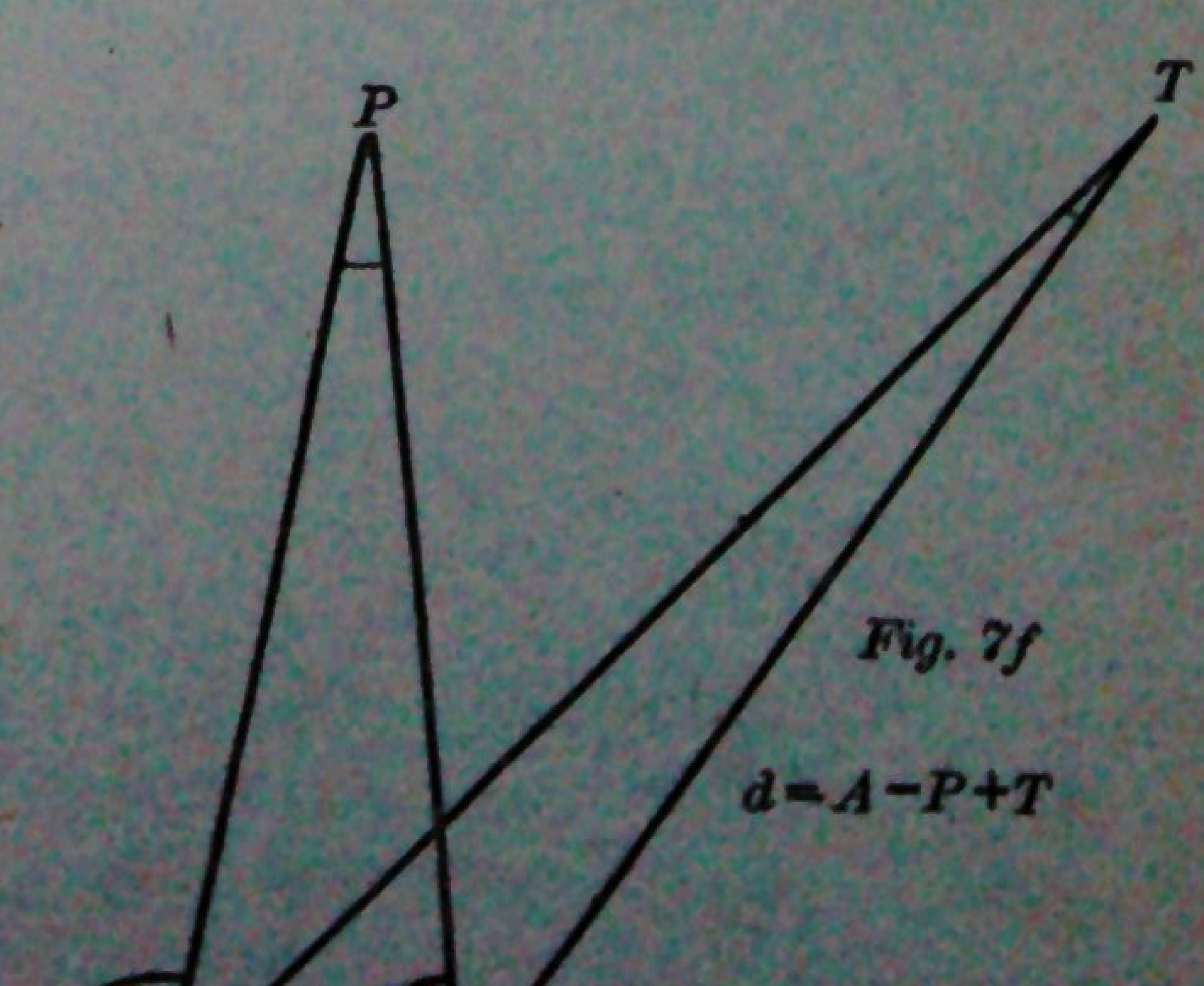



Fig. 7f

$d = A - P + T$

## DETERMINATION OF THE DEFLECTION.

*Accurate Solution*: Since $A$ is given by measurement, the determination of $d$ depends on finding the offsets $P$ and $T$.

In $\triangle PBG$ the angle at $B$ can always be measured, since by assumption both $P$ and $G$ are visible from $B$. Since the lengths $GP$ and $GB$ may also be assumed known, we find $P$ from

$$\frac{\sin P}{GB} = \frac{\sin PBG}{GP}.$$

In $\triangle GBT$ the angle at $B$ can be measured, and the sides $GT$ and $GB$ may be assumed known. Hence

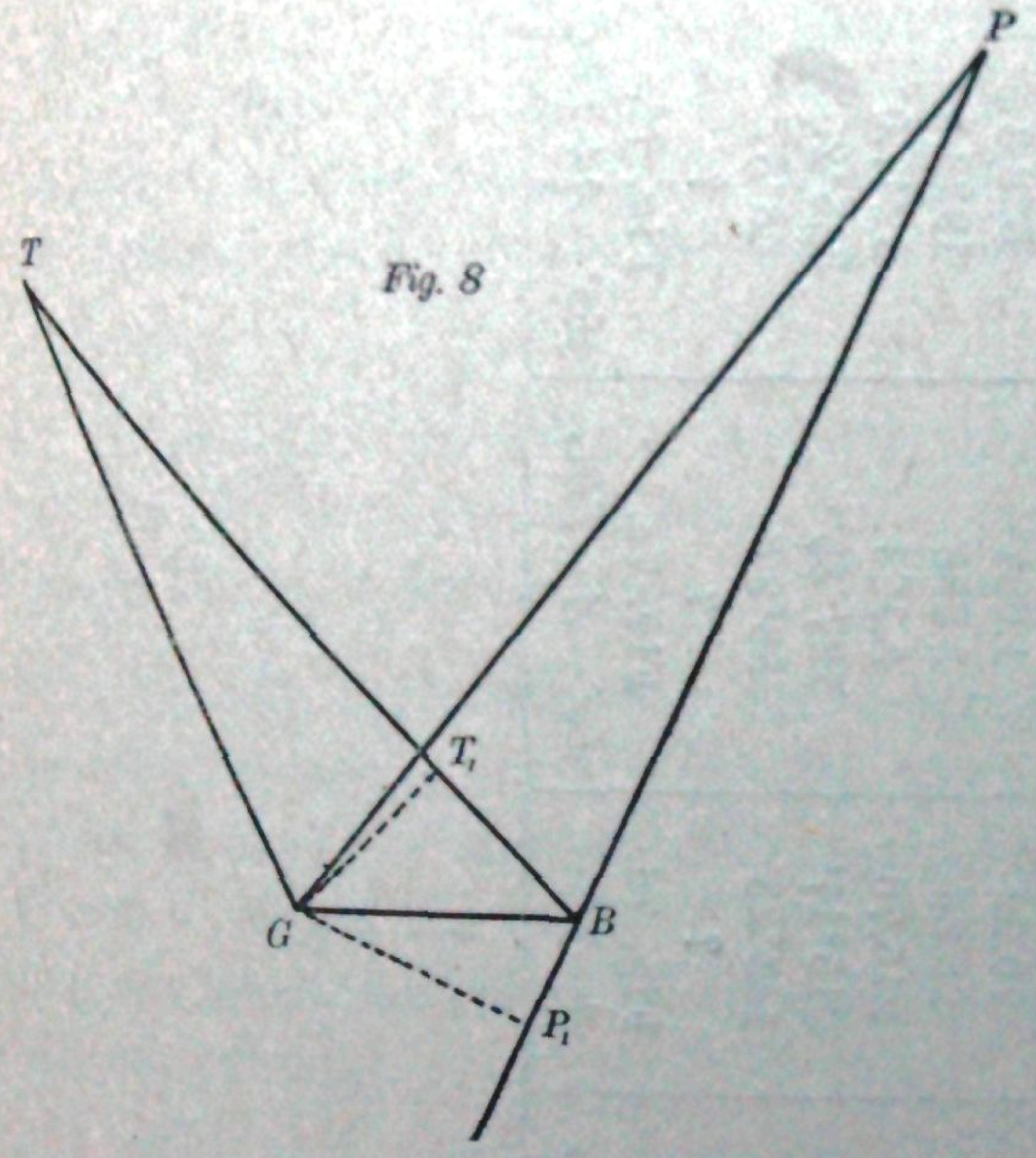
$$\frac{\sin T}{GB} = \frac{\sin TBG}{GT},$$

from which we find $T$. (The three angles: $A$, $TBG$, $PBG$, are not independent of each other, so that it would really be sufficient to measure $A$ and one of the other angles. (See problems of p. 28).

In Field Artillery Service, this method is not employed. Instead, there are several methods of approximation in use which are closely related to each other and which are based on the use of parallaxes. We turn to a brief discussion of one of these methods.

## DETERMINATION OF THE DEFLECTION BY MEANS OF PARALLAXES.

The method of parallaxes can, as we know, be applied with advantage only when we have to deal with lengths of which some are many times longer than others. This is one reason why $B$ is selected close to $G$, while $P$ is selected as distant as possible.




To find $T$, assume the perpendicular $GT_1$ dropped from $G$ onto $BT$ (Fig. 8). Then $GT_1 = GB \cdot \sin GBT$, where both factors on the right side are known. From $\triangle TGT_1$ we find $T$ by the rule of p. 13:

$$T \text{ (in mils)} = GT_1 : \left(\frac{TT_1}{1000}\right)$$

We replace $TT_1$ (which is not known) by $GT$. Since $GT_1$ is small compared with $GT$, the ratio $GT : T_1T$ is nearly unity, so that the new error thus introduced is small. Therefore

$$T \text{ (in mils)} = GT_1 : \left(\frac{GT}{1000}\right).^*$$

In practice $GT_1$ is usually not determined from $GT_1 = GB \cdot \sin GBT$, but is estimated by the battery commander at $B$ from his gnowledge of length $GB$.

$P$ is found in the same way by estimating the length of the perpendicular $GP_1$ from $G$ to $BP$ and determining $P$ from the right triangle $GP_1P$. Thus

$$P = GP_1 : \left(\frac{GP}{1000}\right).$$

In most handbooks on Field Artillery the perpendiculars are dropped from $B$ onto $GT$, $GP$. The subsequent work is practically as above. The arrangement in the text is adopted from Professor Whittemore's article.

---

*This is exactly the method explained in the second solution of the example of p. 15. When $GT_1$ is *estimated*, hardly any computation is required to find $T$. But it is important to note that an *error of, say, s per cent in estimating GT causes an error of s per cent in T.*

## PROBLEMS

This set consists of a few problems to be solved by applying the theorem of sines, as explained on p. 26. While the accurate method of finding the deflection is not used in the Field Artillery, it may give the student a clearer understanding of the background of the theory of indirect firing.

| | 1. | 2. | 3. | 4 | 5. | 6. |
|---|---|---|---|---|---|---|
| *GB* = | 500 | 750 | 415 | 70.32 | 1.087 | 526.3 |
| *GT* = | 1850 | 1875 | 1265 | 95.86 | 3.256 | 1938.0 |
| *GP* = | 2050 | 2425 | 1185 | 215.70 | 3.128 | 1247.0 |
| *A* = | 83° | 315°45′ | 232°25′ | 215°17′ | 127°39′ | 325°26′ |
| *TBG* = | 40°20′ | 132°15′ | 102°35′ | 143°23′ | 86°27′ | 48°37′ |
| *d* = | ? | ? | ? | ? | ? | ? |
| *Type:* | Fig. 7a. | Fig. 7b. | Fig. 7c. | Fig. 7d. | Fig. 7e. | Fig. 7f. |
| *PBG* = | *A*+*TBG* | *A*+*TBG*−360° | 360°−*A*−*TBG* | *A*−*TBG* | 360°−*A*−*TBG* | 360°−*A*+*TBG* |

Since it is necessary to work with the sines of angles in the accurate solution, the angles are given in degrees, because every angle in mils would have to be changed to degrees before tables can be used. Besides, the base $GB$ has been chosen larger in comparison with the range $GT$ than is permissible when the method of parallaxes is employed. This was done in order to avoid triangles with one angle nearly zero. The student is advised to construct the figures.

Problems to be solved by the method of parallaxes have not been inserted, because an artillery officer receives in the army a thorough training in the determination of the deflection by methods of approximation.

*Remarks*: It will be noticed that if the battery commander's station ($B$) is chosen as aiming point, that is, if $B$ and $P$ coincide, IVb yields IVa as a special case.

It might therefore seem an unnecessary complication to choose a separate aiming point. However, the method of approximation explained above (or similar methods) permit a very rapid calculation of $P$ and $T$ within the limits of accuracy required for Field Artillery, so that no appreciable loss of time is involved in choosing for $B$ and $P$ distinct points.

On the other hand, it is desirable to choose the aiming point as distant as possible, while the battery commander's station is generally desired at a moderate distance from the gun. This arrangement is considered best for practical reasons connected with the question of fire control by the battery commander and which arise largely from the fact that the Field Artillery never uses individual guns as a unit, but whole batteries.

In fact, the indirect method of pointing, with a distant aiming point distinct from the battery commander's station, is frequently employed even when the target is visible from the guns, on account of the advantage of centralized fire control.

Correction: p. 24 read Rule:
*One obtains, however B and P are chosen, a relation of the type*

$$d = A \pm P \pm T.$$