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# MATHEMATICAL MODELS

## IV SERIES

BY

ARNOLD EMCH

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Department of Mathematics

1928
UNIVERSITY OF ILLINOIS
URBANA

# PREFACE

The mathematical models listed and described below have been planned and designed in the mathematical laboratory of the University of Illinois since the publication of the first series in 1921, the second series in 1923, and the third series in 1925. As has been stated in the first three series, the purpose of these constructions is to represent certain features of mathematical instruction and research by adequate models, mechanisms, or graphs, which are not available in the market.

In this new series various types of illustrative mathematical technique are described.

The first is a series of lantern slides for use in lectures on algebraic geometry, in particular on algebraic curves. The figures of these slides have been selected with reference to their importance in the illustration of the theory.

Next comes a set of three models on perspective, Desargues theorem, and affinity, for use in a course on projective geometry.

Series II contains illustrations and short descriptions of two space-sextics of genus four. In the new series is added a third model of this type showing the possibility of 120 real tritangent planes of such a sextic. There is also a model of the Weddle surface, constructed by W. L. Moore in connection with a study of the geometry on this surface.

Differential geometry is represented by two models for the developability of surfaces of constant curvature.

Parties or institutions interested in these models or any of those of series I, II, III, may procure duplicates by making arrangements with local private firms for the manufacture and sale of such duplicates.

Information concerning this and possibly other questions on models will be given by Arnold Emch, Professor of Mathematics, University of Illinois.

Urbana, Illinois.
January, 1928.

# MATHEMATICAL MODELS

## IV Series[a]

## A. CLASSIFICATION OF PLANE CUBICS

Newton in his *Enumeratio linearum tertï ordinis*,[b] first published in 1704, classified plane curves of the third order into 5 types according to the values of $\alpha$, $\beta$, $\gamma$ in the equation

$$y^2 = a\,(x-\alpha)\,(x-\beta)\,(x-\gamma):$$

I. $\alpha < \beta < \gamma$, and all real. Cubics consisting of an infinite branch and an oval = bipartite cubics.

II. $\beta$ and $\gamma$ are conjugate complex. Results from III. by vanishing of isolated point.

III. $\beta = \gamma \gtrless \alpha$. Same as I., but oval shrinks to isolated singularity.

IV. $\alpha = \beta \gtrless \gamma$. Obtained from I., when infinite branch and oval join across a double point.

V. $\alpha = \beta = \gamma$. Cubic with cusp.

As the line at infinity is a tangent to the cubic in the Newtonian form, the invariant formed by the cross-ratio of the four tangents from the infinite point of the y–axis to the cubic becomes $(\alpha\beta\gamma\infty) = (\gamma-\alpha)/(\gamma-\beta)$ and the absolute invariant

$$J = 24\frac{(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)^3}{[2(\alpha^3+\beta^3+\gamma^3)-3\{\alpha\beta(\alpha+\beta)+\beta\gamma(\beta+\gamma)+\gamma\alpha(\gamma+\alpha)\}+12\alpha\beta\gamma]^2}.$$

As this may assume any value, and as there are $\infty^1$ projectively different cubics, all cubics may be obtained projectively from the Newtonian five types. This was pointed out by Newton who made the statement that all cubics may be obtained from the five types by projection (shadows).

A very effective constructive treatment of the five types may be obtained by making use of the fact that cubics are invariant in certain involutorial quadratic transformations, as is shown by the author.[c]

The Steinerian transformation in its general form may be based upon a pencil of conics $(ax)^2 + \lambda\,(bx)^2 = 0$ in a plane $(x)$, so that to

[a]The last number of series III was 31. In this, as in prospective series, the models will be numbered continuously, so that in any inquiry concerning these models it will suffice to state the corresponding number.

[b]Isaaci Newtoni Opera quae extant omnia, commen, S. Horsley, London 1779-1785, vol. 1, pp. 531-560.

[c]*Newton's five types of plane cubics obtained by the Steinerian transformation.* The University of Colorado Studies, vol. 1, pp. 275-284 (1904).

a point $P$ $(y)$ in $(x)$ corresponds the vertex $\{(ay)(ax) = 0, (by)(bx) = 0\}$, or $P'$ $(y')$, of the pencil of polars $(ay)(ax) + \lambda (by)(bx) = 0$ of $(y)$ with respect to the pencil of conics. $P$ and $P'$ correspond to each other in an involutorial quadratic Cremona transformation, which has the base-points $A_1A_2A_3A_4$ of the pencil of conics as invariant points and the diagonal triangle $B_1B_2B_3$ of the quadrangle $A_1A_2A_3A_4$ as the fundamental triangle. If we choose the latter as the coordinate triangle and $A_4$ as the unit point, such that $A_1A_4$ and $A_2A_3$, $A_2A_4$ and $A_3A_1$, $A_3A_4$ and $A_1A_2$ intersect in $B_1B_2B_3$ respectively, the transformation may be put in the standard form.

$$\rho x_1' = x_2x_3, \ \rho x_2' = x_3x_1, \ \rho x_3' = x_1x_2.$$

When $A_4$ is chosen as the orthocenter of the triangle $A_1A_2A_3$ the construction of the transformation becomes particularly simple. Let $P$ be a generic point. Join $P$ to $B_1$ and $B_2$ and construct the line $l_1$ symmetric to $PB_1$ with respect to $B_1A_4$, $l_2$ symmetric to $PB_2$ with respect to $B_2A_4$ as an axis. Then $l_1$ and $l_2$ intersect in the point $P'$, which corresponds to $P$ in the transformation. As a check for the construction the symmetric $l_3$ to $PB_3$ with respect to $B_3A_4$ must also pass through $P'$, Fig. 1.

This transformation was known to Poncelet,[a] and also to Steiner[b] who defined it as the correspondence between points which are simultaneously harmonic with respect to the three pairs of lines of a quadrangle, which, with $B_1B_2B_3$ as the vertices of these pairs, are $x_2 - x_3 = 0$, $x_2 + x_3 = 0$; $x_3 - x_1 = 0$, $x_3 + x_1 = 0$; $x_1 - x_2 = 0$, $x_1 + x_2 = 0$. The designation "Steinerian" for this transformation was introduced by Durège[c] and adopted by Schröter,[d] and others.

The procedure by which the theory of cubics may be connected with the Steinerian transformation is as follows: To a line $l$ corresponds a conic $L$ which cuts $l$ in a couple of corresponding points $P$, $P'$. These are also the double points of the involution cut out on $l$ by the pencil of conics through $A_1A_2A_3A_4$, or the points of contact with $l$ of two definite conics of the pencil. Conversely, with every couple $P$ $(x_1, x_2, x_3)$, $P'$ $(x_2x_3, x_3x_1, x_1x_2)$ is, in general, uniquely associated a line $l$ whose coordinates are $\rho u_1 = x_1(x_2^2 - x_3^2)$, $\rho u_2 = x_2(x_3^2 - x_1^2)$, $\rho u_3 = x_3(x_1^2 - x_2^2)$.

The locus of couples of corresponding points on the tangents of a curve $K_m$ $(u_1, u_2, u_3) = 0$ is in general a curve of order $3m$, which is invariant in the transformation. When $K_m$ is rational then the invariant $C_{3m}$ becomes hyperelliptic. The theory of these invari-

[a] Traité des propriétés projectives des figures (1822), 2nd ed. 1865, vol. 1, pp. 185-248.
[b] J. f. Math. (Crelle's), vol. 3 (1828), pp. 207-212; Werke, vol. 1, pp. 173-180.
[c] Die ebenen Curven dritter Ordnung (1871) pp. 121-128.
[d] Steiner - Schröter; Vorlesungen über syn'thetische Geometrie, vol. 2, 2nd ed. (1876), pp. 301-304.

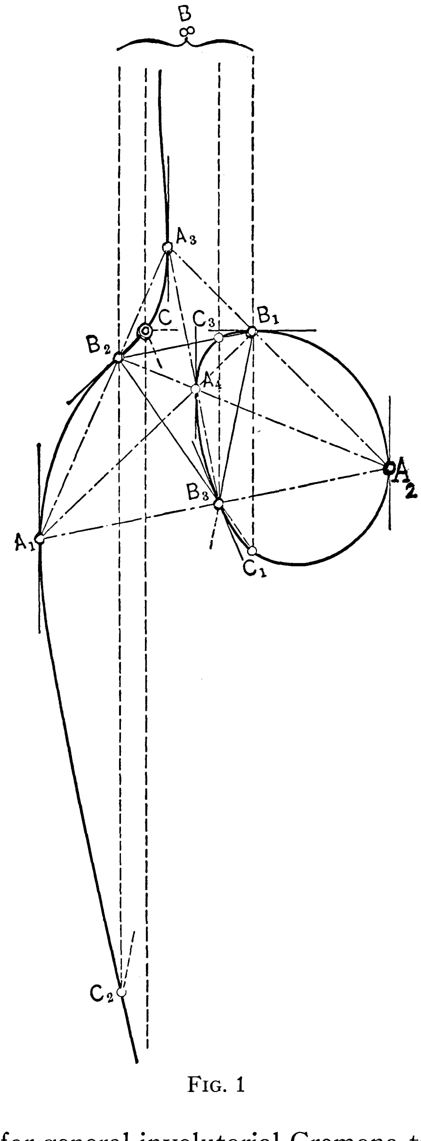


FIG. 1

ant curves, also for general involutorial Cremona transformations in the plane, and with extensions to space, has been given by the author in a number of papers.[a]

Now when $K_m$ is a point $(a_1a_2a_3) = (a) = B$

$$a_1u_1 + a_2u_2 + a_3u_3 = 0,$$

we get the invariant cubic

[a]See *On surfaces and curves which are invariant under involutory Cremona transformations*, American Journal of Mathematics vol. 48 (1926), pp. 21-44, and references given.

$$a_1x_1\,(x_2{}^2 - x_3)^2 + a_2x_2\,(x_3{}^2 - x_1{}^2) + a_3x_3\,(x_1{}^2 - x_2{}^2) = 0,$$

which, with De Jonquières, we call the isologue of $(a)$, and which passes through $A_1A_2A_3A_4$ and $B_1B_2B_3$. The joins $BA_1$, $BA_2$, $BA_3$, $BA_4$ are tangents to the cubic at $A_1A_2A_3A_4$. This is a socalled Steinerian quadruple of the cubic. Moreover $BB_1B_2B_3$ is also such a quadruple, i.e., the tangents to the cubic at $BB_1B_2B_3$ are concurrent at a point $C$. The Steiner transformation attached to the quadruple $B$ $B_1B_2B_3$ leaves the cubic also invariant. In this second transformation to $C$ corresponds a point $C'$ which, together with the diagonal triangle $C_1C_2C_3$ of $BB_1B_2B_3$ forms a new Steinerian quadruple. This process may be continued indefinitely and thus furnishes the construction of an unlimited number of points of the cubic. In the figure $B$ is the infinite point of the serpentine.

The different Newtonian types may be obtained by this method by enumerating the various possibilities for the base-points $(A)$ of the pencil of conics, or for the intersection of two conics.

## I. THE CUBIC SERPENTINE WITH OVAL

This cubic is obtained when all four points of the fundamental triangle are either real, or imaginary. As the case of four real points is illustrated in Fig. 1, I shall now assume an entirely imaginary quadruple which is determined by the imaginary fundamental points of an elliptic coaxial system of circles and the circular points at infinity through which it also passes. Let $P$ and $Q$ be the limiting points of the system, Fig. 2.

$P$, $Q$ and the infinite point of $m$ form the fundamental triangle of the associated Steiner transformation. On every ray $g$ through an arbitrary fixed point $B$ the circles of this system cut out an involution of points whose double-points $X$ and $X'$ are two points of the circular cubic associated with the point $B$ in the Steinerian transformation of the given imaginary quadruple. The points $X$ and $X'$ are also the points of tangency of $g$ with two circles of the given coaxial system. Hence, according to a well known construction, the points $X$ and $X'$ are obtained by finding the point of intersection $M$ of $g$ with $m$, the line joining the finite imaginary points of the quadruple. With $M$ as a center pass a circle $K$ through $P$ and $Q$ which will cut $g$ in the required points. From the figure it is seen that the two points of the cubic on a ray through $B$ are equally distant from $m$. Hence, taking a ray through $B$ parallel to $m$, the point at infinity corresponding to $E$ will be in a line $a$ through $C$, which corresponds

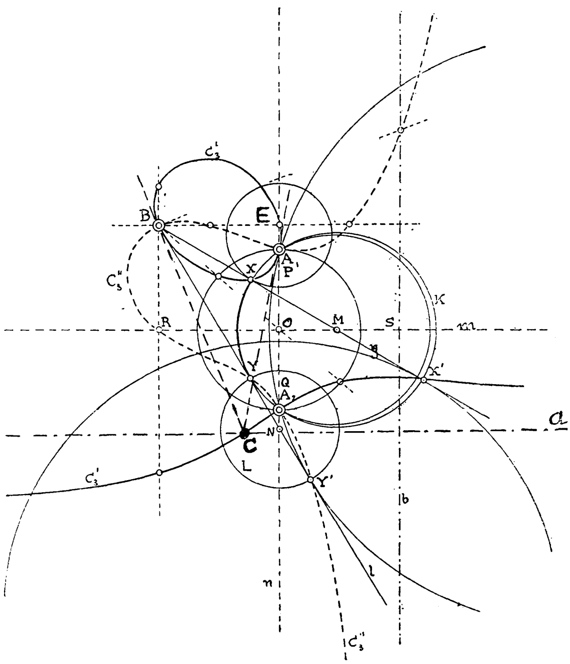


FIG. 2

to $B$, parallel to $m$. The tangents at $B$, $P$, $Q$ and the real infinite point of the cubic meet at $C$. In other words, the line $a$ is the real asymptote of the cubic. Considering the pencil of circles through $P$ and $Q$, the same circular cubic is also produced by this pencil and the pencil of corresponding diameters through $B$.

## II. THE CUBIC SERPENTINE

This curve is produced by assuming two separate real and two conjugate imaginary points as the fundamental quadruple. In Fig. 2, let $A_1$ and $A_2$ be the real points and the circular points of the pencil of circles through $A_1$ and $A_2$ the imaginary points. To find the points $Y$ and $Y'$ where a ray $l$ through $B$ cuts the cubic, let $l$ cut $n$ at $N$. With $N$ as a center construct the circle $L$ orthogomal to the pencil of circles through $A_1$ and $A_2$, Fig. 2. The circle cuts $l$ in the required points $Y$ and $Y'$. This cubic appears, again, plainly as the product of a pencil of circles and a pencil of diameters through $B$. Two points $Y$ and $Y'$ on a ray through $B$ are always equally distant from $n$. To $R$ corresponds the infinitely distant point of the cubic; consequently the asymptote $b$ is parallel to $n$ and its distance $SO$ from $m$ is equal to $RO$.

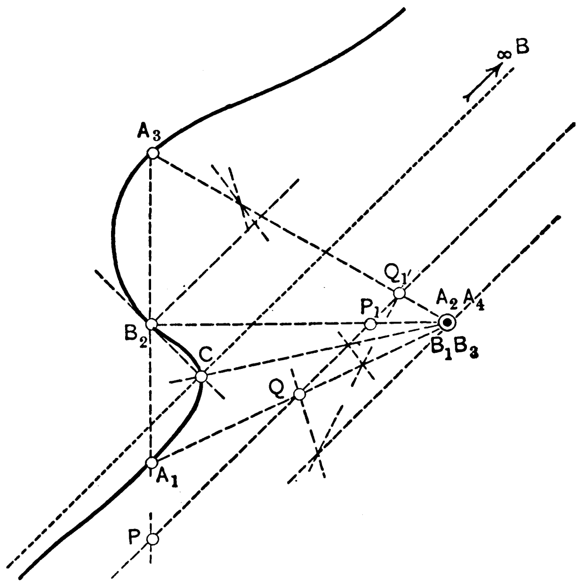


FIG. 3

## III. THE CUBIC SERPENTINE WITH ISOLATED POINT

The quadruple consists of two distinct points $A_1A_3$ and two coincident points $A_2A_4$. It is assumed that the direction of the line joining $A_2$ with $A_4$ in the limit, i.e., as they become coincident, cuts $A_1A_3$ at $B_2$. $B_1$ and $B_2$ coincide with $A_2A_4$, Fig. 3. In the Steinerian transformation we find the point $C$ corresponding to a point $B$, by joining $B$ to $B_1$, $B_2$, $B_3$ and constructing the fourth harmonic rays to these joining lines with respect to the pairs of sides of the quadruple through the points $B$. The three fourth harmonic rays concur at the required point $C$. In our case the rays $B_1C$ and $B_3C$ coincide, as can easily be seen by passing over to the limit. As in the general case of a real quadruple, they cut the fourth harmonic ray through $B_2$ at $C$, the point through which the asymptote passes. The pencil of conics through the quadruple cuts every ray through $B$ to the left of $A_3$ and the right of $A_1$ in elliptic involutions, and only the rays between $A_1$ and $A_3$ contain hyperbolic involutions. The only branch of the cubic is therefore contained between two lines through $A_1$ and $A_3$ parallel to the direction of $B$. The ray through $A_2A_4$ carries a parabolic involution and $A_2A_4$ represents an isolated point of the cubic.

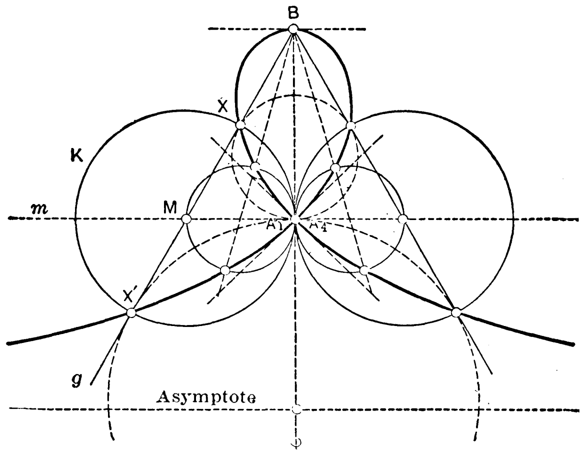


FIG. 4

## IV. THE NODAL CUBIC

Assuming in the fundamental quadruple $A_1$ and $A_4$ real and coincident and $A_2$, $A_3$ conjugate imaginary, a cubic with a double-point, or node, at $A_1A_4$ arises. In Fig. 4, a vertical line through $A_1A_4$ represents the limiting direction of the line joining the two points. As conics of the pencil through the fundamental quadruple take the pencil of circles tangent to each other at $A_1A_4$ and to the vertical line. $A_2$ and $A_3$ are then represented by the circular points at infinity. To construct the cubic associated with an arbitrary point $B$, draw rays through $B$. On each of these rays the pencil of circles cuts out an involution whose double-points are points of the cubic. These points are also the points of tangency of circles of the pencil. Hence, to find the points where a ray $g$ through $B$ cuts the cubic, take the point $M$ where $g$ cuts $m$ as a center of a circle $K$ passing through $A_1A_4$. $K$ cuts $g$ in the required points $X$ and $X'$. From this it is seen that this cubic is also the product of a pencil of circles with coincident limiting points and a pencil of diameters through $B$. As $X$ and $X'$ are equally distant from $m$, the asymptote is parallel to $m$ at a distance to the left of $m$ equal to $BA_1$ ($BA_1 \perp m$ for the sake of symmetry).

## V. THE CUSPIDAL CUBIC

In this case three of the four points of a real fundamental quadruple coincide. Constructively such an arrangement can be realized by assuming as the pencil of conics a pencil through a fixed point

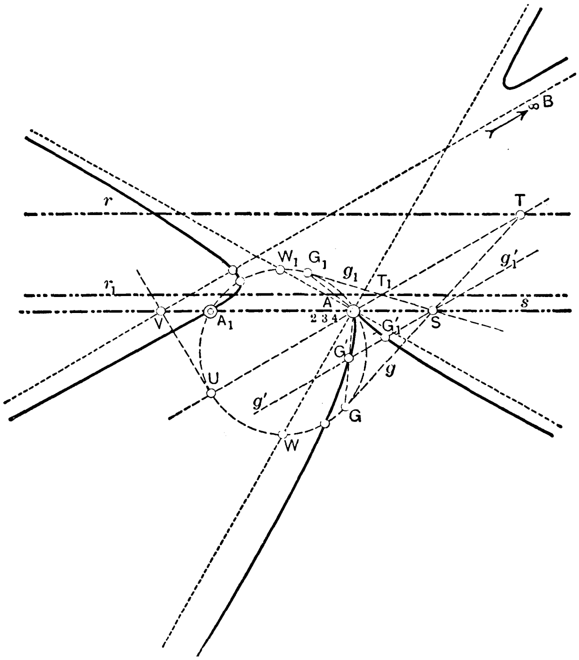


FIG. 5

$A_1$ and with its conics osculating each other at another fixed point which evidently may be considered as the representative of the three coincident points $A_2A_3A_4$.

To construct a pencil of osculating conics we may start with the fact that the picture of a circle in a perspective collineation whose center lies on the axis of collineation and also on the given circle is a conic osculating the given circle at the center of collineation.[a] Hence, considering in Fig. 6[b] the line $s$, joining $A_1$ with the coincident remaining points, as the common axis of an infinite number of perspective collineations in which only the counter-axes[c] vary, then the pictures of a fixed circle $K$ through $A_1A_2A_3A_4$ clearly form a pencil of osculating conics.

On every ray $g'$ (or the identical $g_1'$) through a fixed point $B$ (assumed infinitely distant) this pencil cuts out an involution whose double-points are two points on the cuspidal cubic associated with $B$ in the Steinerian transformation. These points are also the points

[a]Fiedler: Darstellende Geometrie, Vol. I (3rd ed.), pp. 188-190.
[b]The branch of the cubic on the upper right side has only been indicated. In the construction it fell beyond the border of the figure.
[c]See Fiedler, loc. cit., pp. 47-49.

of tangency of $g'$ ($g_1'$) with two conics of the pencil. For the actual construction the following simple method may be applied. Let $g'$ intersect $s$ at $S$. From $S$ draw the two tangents $g$ and $g_1$ to the circle $K$. Through the center of collineation (cusp) draw a line $l$ parallel to the direction of $B$. Let $T$ and $T_1$ be the points of intersection of $l$ with $g$ and $g_1$, and through $T$ and $T_1$ draw two lines $r$ and $r_1$ parallel to $s$. Considering $r$ and $r_1$ as counter-axes of two collineations with the same axis $s$ and the same center, then, according to the constructions of collineation, $g'$ and $g_1'$ are the pictures of $g$ and $g_1$ in these two collineations, and the rays joining $C$ to $G$ and $G_1$ cut $g'$ ($g_1'$) in two points $G'$ and $G_1'$ which evidently are the points of tangency with $g'$ ($g_1'$) of the two osculating conics corresponding to $K$ in the two collineations ($r$, $r_1$). The line $l$ cuts $K$ at $U$; the tangent at $U$ cuts $s$ at $V$, and from the construction follows that the line through $V$, parallel to $l$, is the direction of an asymptote. In similar manner the lines joining $C$ to the points of tangency $W$ and $W_1$ of the tangents to $K$, parallel to $s$, are the directions of the asymptotes.

By proper collineations it is not difficult to transform the five cubics constructed by means of the Steinerian transformation into Newton's five symmetrical types.

More detailed classifications were given by Murdoch,[a] Möbius,[b] Cayley.[c]

F. Kölmel[d] from a purely algebraic standpoint and H. Wiener[e] by purely geometric methods find a complete classification into 13 types based upon the values of $\lambda$ in Hesse's normal form

$$x_1^3 + x_2^3 + x_3^3 + 6\lambda x_1 x_2 x_3 = 0.$$

In the graphic representation of real cubics account must be taken of the behavior of these curves at infinity. The possible various shapes may be obtained from the Newtonian five types by perspective, for example by the involutorial perspective.

$$x' = \frac{x}{y-1}$$

$$y' = \frac{y}{y-1}$$

[a] *Genesis Curvarum per Umbras*, London 1746.

[b] *Über die Grundformen der Linien dritter Ordnung*, 1852, Ges. Werke, vol. 2, p. 90.

[c] *On the Classification of Cubic Curves*, Camb. Phil. Soc. Trans., Vol. II, pp. 81-128 (1865). *On Cubic Cones*, same volume pp. 129-144.

[d] *Ableitung der verschiedenen Formen der Kurven dritter Ordnung durch Projektion mid Klassifikation derselben*, I. progr. Ettenheim, 12 pp. (1894); II. progr. Mosbach, 12 pp. (1895); III. progr. Baden-Baden, 14 pp. (1904).

[e] *Die Einteilung der ebenen Kurven und Kegel dritter Ordnung in 13 Gattungen.* Halle, A. S. Schilling, 34 pp. (1901).

The axis of perspective (point-wise invariant) is the line $y = 0$. To the line $r$ ($y = 1$) corresponds the line at infinity and to the later the line $q'$ ($y = 1$), so that $r \equiv q'$. By choosing the line $r$ in distinctly different ways with respect to the cubic, we obtain by the above involution the various forms of cubics. Suppose $r$ cuts the cubic in the points $S_1$, $S_2$, $S_3$, and let $\alpha_1$, $\alpha_2$, $\alpha_3$ be the tangents to the cubic at these points. Then the projections $\alpha_1'$, $\alpha_2'$, $\alpha_3'$ of $\alpha_1$, $\alpha_2$, $\alpha_3$ are the asymptotes, of the projected cubic. These may all be real, one real and two conjugate complex, one distinct and two coincident, all three coincident. Thus when $d_1$ in a flex-tangent, $d_1'$ becomes a flex-asymptote.

The list given in Figures 6, 7, 8, 9, and 10 is obtained in this manner.

*Lantern slides for screen-projections of classified cubics.*

**32.** The five Newtonian types by the Steiner transformation.

**33.** Types of cubics in the graphic classification (including the five Newtonian types.

## I. SERPENTINE CUBICS (UNIPARTITE)[a]

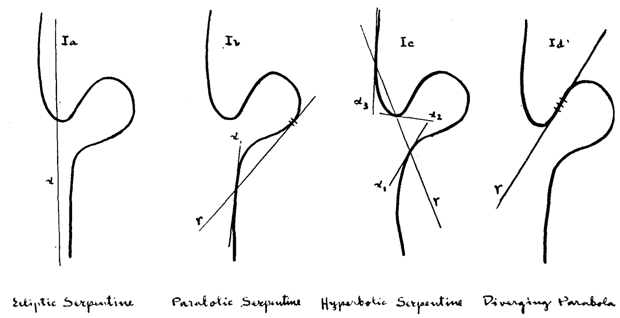


Fig. 6

$I_a$ *Elliptic serpentine.* One real asymptote.
$I_b$ *Parabolic serpentine.* One real asymptote. Touches line at infinity.
$I_c$ *Hyperbolic serpentine.* Three real asymptote.
$I_d$ *Diverging parabola.* Has line at infinity as flex-tangent.
$I_e$ *Serpentine with three real flex-tangents.*

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[a]In this list the orders of I and II are those of II and I given above, in order to have agreement with the list given in Wieleitner's Algebraische Kurven.

## II. CUBIC, SERPENTINE WITH OVAL (BIPARTITE)

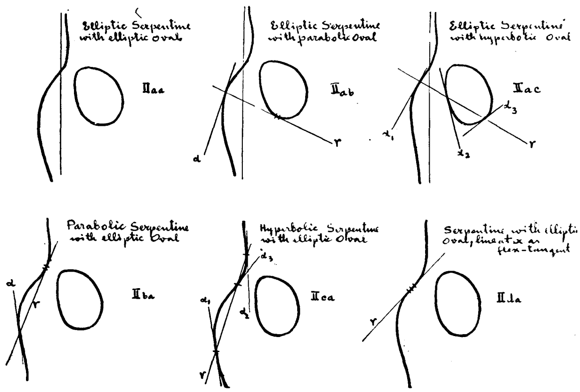


FIG. 7

$II_{aa}$ *Elliptic serpentine with elliptic oval.* One real asymptote.

$II_{ab}$ *Elliptic serpentine with elliptic oval.* One real asymptote.

$II_{ac}$ *Elliptic serpentine with hyperbolic oval.* Three real asymptotes.

$II_{ba}$ *Parabolic serpentine with elliptic oval.* One real asymptote.

$II_{bc}$ *Hyperbolic serpentine with elliptic oval.* Three real asymptotes.

$II_{da}$ *Serpentine with elliptic oval;* line at infinity as flex-tangent.

$II_{e}$ *Serpentine with oval* with three real flex-asymptotes.

## III. CUBIC WITH ISOLATED DOUBLE-POINT J.

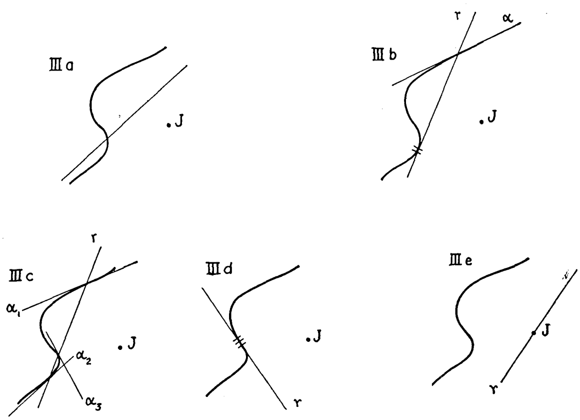


FIG. 8

$III_a$ *Elliptic serpentine with isolated double-point.* One real asymptote.

$III_b$ *Parabolic serpentine with isolated double-point.* One real asymptote.

$III_c$ *Hyperbolic serpentine with isolated double-point.* Three real asymptotes.

$III_d$ *Serpentine with line at infinity as flex-tangent, with isolated double-point.*

$III_e$ *Serpentine with isolated double-point at infinity.* One real asymptote.

## IV. NODAL CUBICS

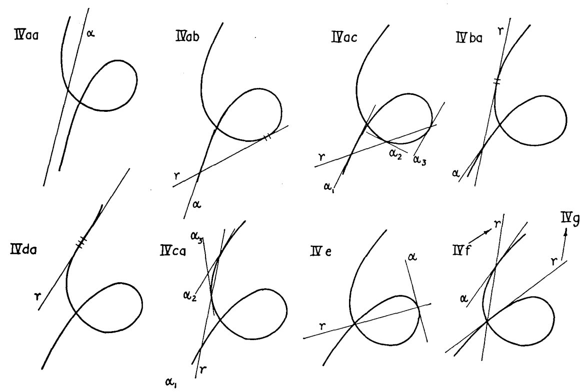


FIG. 9

IV$_{aa}$ *Nodal cubic with one real asymptote.*

IV$_{ab}$ *Nodal cubic with one real asymptote and loup touching line at infinity.*

IV$_{ac}$ *Nodal cubic with three real asymptotes.*

IV$_{ba}$ *Nodal cubic with one real asymptote and open branch touching line at infinity.*

IV$_{ca}$ *Nodal cubic with three real asymptotes.*

IV$_{da}$ *Nodal cubic with line at infinity as flex-tangent.*

IV$_{e}$ *Cubic with node at infinity and line at infinity cutting loop.*

IV$_{f}$ *Cubic with node at infinity and line at infinity cutting open branch.*

## V. CUSPIDAL CUBICS

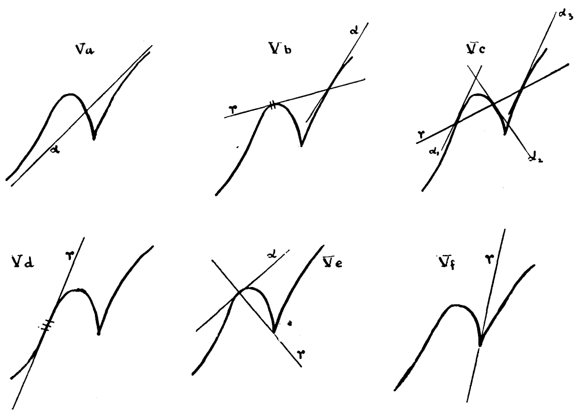


FIG. 10

$V_a$ *Cuspidal cubic with one real asymptote.*

$V_b$ *Cuspidal cubic with one real asymptote and tangent to line at infinity.*

$V_c$ *Cuspidal cubic with three real asymptotes.*

$V_d$ *Cuspidal cubic with line at infinity as flex-tangent.*

$V_e$ *Cubic with cusp at infinity.*

$V_f$ *Cubic with line at infinity as cuspidal tangent.*

## B. THEORY OF CUBICS

### LANTERN SLIDES FOR SCREEN-PROJECTIONS

**34.** *Dual Singularities.*

The $C_3$ with node at $D$ is a $\Gamma_4$ of class 4. To the tangents of $\Gamma_4$ correspond in the polar correlation with respect to the circle $C$ (center $O$) the points of a $C_4$. To the double-point $D$ $(a, b)$ corresponds the double tangent $d$ $(A, B)$; to the inflexion $I$ $(f)$ the cusp $i$ $(F)$, Fig. 11.

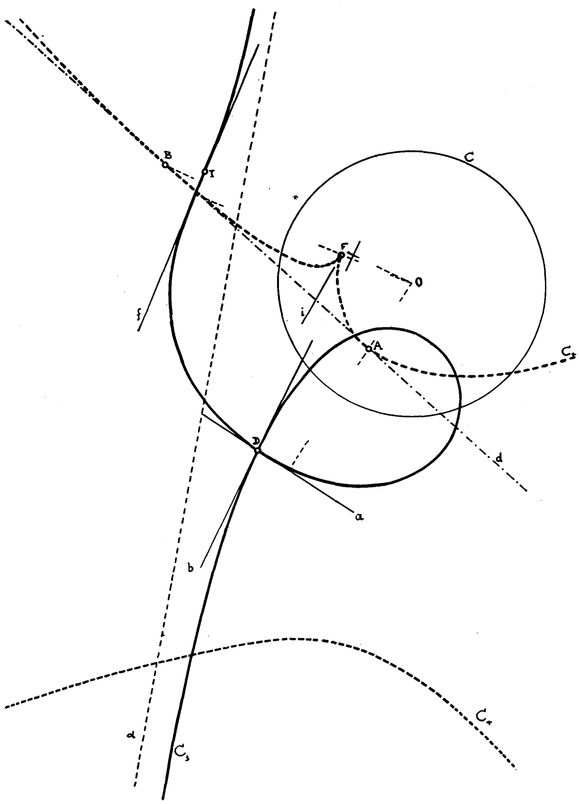

FIG. 11

**35.** *Self Dualistic Cuspidal Cubics.*

To the cubic $C_3$ with cusp $K\ (f)$ at $K$ and inflexion $I\ (\varphi)$ at $I$ corresponds in the polar correlation, with respect to the circle $C$ (center $O$), the cubic $\Gamma_3$ with inflexion $k\ (F)$ corresponding to the cusp $K\ (f)$, and the cusp $i\ (\phi)$ corresponding to the inflexion $I\ (\varphi)$, Fig. 12.

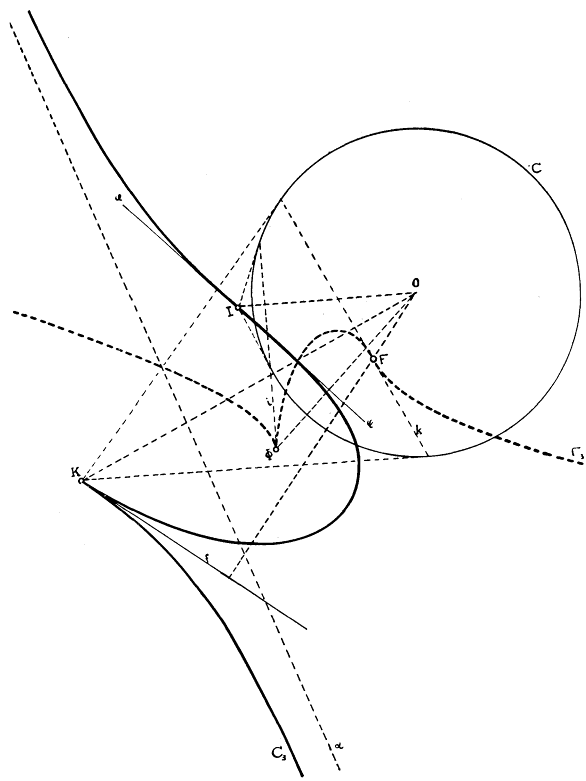


FIG. 12

**36.** *Equianharmonic Cubics.*

$$E_1 = x_1x_2^2 + x_2x_3^2 + x_3x_1^2 = 0.$$
$$E_2 = x_1x_3^2 + x_2x_1^2 + x_3x_2^2 = 0.$$

For both $E_1$ and $E_2$ the Hessian $H$ is the same

$$H = x_1^3 + x_2^3 + x_3^2 - 3x_1x_2x_3 =$$
$$(x_1 + x_2 + x_3)(x_1 + \epsilon x_2 + \epsilon^2 x_3)\ (x_1 + \epsilon^2 x_2 + \epsilon x_3) = 0.$$

The real flexes are on the unit-line $x_1 + x_2 + x_3 = 0$, and the flex-tangents at the real flexes all meet at $E$ (111). $E_1 + \lambda E_2 = 0$ is a pencil of equianharmonics, Fig. 13.

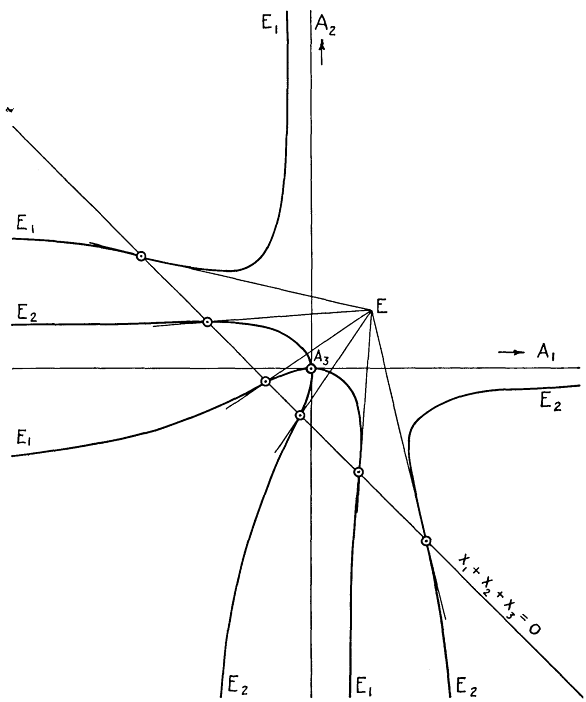


FIG. 13

**37.** *Harmonic Cubic.*

$$C \equiv y^2 - x^3 + 4x = 0.$$

The Hessian is $H \equiv 12x^2 - 3xy^2 + 16 = 0.$
The Hessian of $H$ is $C$ itself, Fig. 14.

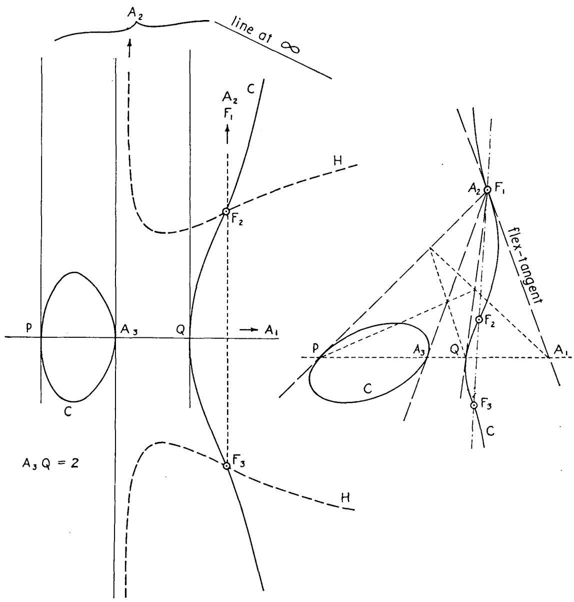


Fig. 14

**38.** *Cubic Invariant in Involutorial Quadratic Transformation.*

$C \equiv x_1(x_2^2 + x_3^2) + x_2(x_3^2 + x_1^2) + x_3(x_1^2 + x_2^2) + \lambda x_1 x_2 x_3 = 0.$
is invariant in $\rho x_1' = x_2 x_3$, $\rho x_2' = x_3 x_1$, $\rho x_3' = x_1 x_2$.
$P$ and $P'$ are corresponding points, also a couple on cubic considered as Hessian = Steinerian of three other cubics. The join of $P$ and $P'$ envelopes Cayleyan class-cubic, Fig. 15.

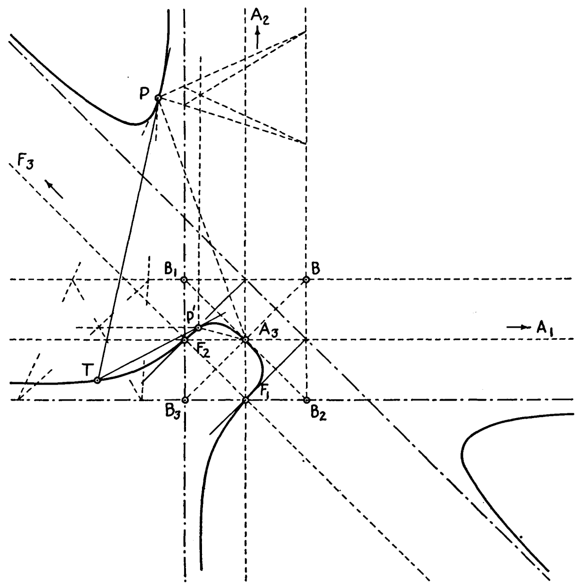


FIG. 15

$$u_1^3 + u_2^3 + u_3^3 - u_1(u_2^2 + u_3^2) - u_2(u_3^2 + u_1^2) - u_3(u_1^2 + u_2^2) + \lambda u_1 u_2 u_3 = 0.$$

$F_1$, $F_2$, $F_3$ are real flexes. $BB_1B_2B_3$ are here invariant points and $A_1A_2A_3$ fundamental points of transformation.

**39.** *Relation Between Cubic and its Hessian and Cayleyan.*
Choosing the cubic in the normal form

$$F \equiv x_1^3 + x_2^3 + x_3^3 + 6m\, x_1x_2x_3 = 0,$$

the Hessian is

$$H \equiv x_1^3 + x_2^3 + x_3^3 + 6\mu x_1x_2x_3 = 0,$$

in which $6\mu = -(1 + 2m^3)/m^2$.
From this is seen that for a given Hessian there are three cubics $F$ for which the given $H$ is the Hessian. As is well known the Hessian

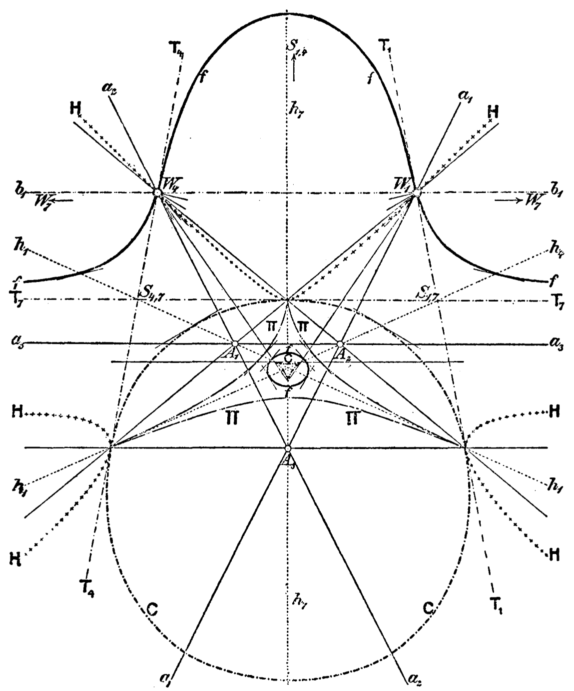

FIG. 16

is the locus of points $P$ whose conical polars degenerate into pairs of lines. The locus of the vertices $P'$ of these pairs coincides with $H$. The joins of $P$ and $P'$ envelope the Cayleyan, Fig. 16.

$$C \equiv m\,(u_1^3 + u_2^3 + u_3^3) + (1 - 4m^3)\, u_1u_2u_3 = 0$$

of class 3. It bears the same relation to the sides of the coordinate triangle as $H$ does with respect to the vertices. $H$ touches the flex-tangents of $F$ and the harmonic polars of the flexes of $F$ are the cuspidal tangents of $C$. Thus there exists duality between flexes, flex-tangents of $F$ and cuspidal tangents (harmonic polars), and cusps of $C$. According to this dualism we can start from a class-cubic $\Pi$ whose 9 cusps are at the points of tangency of $H$, or $C$, with the flex-tangents of $F$, and whose cuspidal tangents coincide with the harmonic polars of $F$. $C$ is now the Hessian of $\Pi$, and $H$ of $F$ the Cayleyan of $\Pi$.[a]

**40.** *Problems of Closure on a Cubic.*

In the cubic $II_{aa}$ let $A_1A_2A_3A_4$ be again be the invariant quadrangle of a Steinerian transformation,[b] and $B_1B_2B_3$ the fundamental triangle.

Let the point $B$ to which the invariant cubic is attached as an isologue be at infinity in the direction indicated. $A_1A_2A_3A_4$ forms a Steinerian quadruple, and the tangents at these points are concurrent at $B$. Likewise $BB_1B_2B_3$ form a Steinerian quadruple with point of concurrence of tangents at $C_0$. The line through $C_0$ parallel to the direction of $B$ is the real asymptote of the cubic. The cubic may be generated in an infinite number of ways by projective quadratic pencils around the points of a Steinerian couple (points with the same tangential) as vertices. Thus around $B$ and $B_2$, $ax$ and $a'x$, and $bx$ and $b'x$ are two pairs of such a projectivity, intersecting in two pairs of corresponding points $X$, $Y$ and $X_1$, $Y_1$ of the Steinerian transformation. Moreover $X$, $X_1$ and $Y$, $Y_1$ form two Steinerian couples. If we join any point on the cubic to a couple, say $X$ to $B_1$ and $B_3$, we get another couple $X'$, $X_1'$ whose tangents have the same point of concurrence on the cubic. Through any Steinerian couple, say $Y$, $Y_1$ there are an infinite number of inscribed quadrilaterals. Thus, join $Y$ to any point ($Y'$) of the cubic. The join will cut the cubic in another point ($B_1$). Join this point to $Y_1$ and find intersection with cubic at a point ($Y_1'$). The join of this and $Y$ will cut the cubic at a point ($B_3$). The join of this point and $Y_1$ will cut the cubic at a point which will coincide with $Y'$. On the figure it can also easily be verfied that the 16 joins of two Steinerian quadruples pass four by four through another Steinerian quadruple, Fig. 17.

The given cubic may be considered as the Hessian of three other cubics. The three pairs of points which may be formed with a Steinerian quadruple of the given cubic are couples of corresponding points $P$ and $P'$ on the Hessian and Steinerian of those three cubics respectively. They belong to three distinct involutions of

[a]See H. Wieleitner, *Theorie der algebraischen Kurven höherer Ordnung*, pp. 232-234 (1905).
[b]Arnold Emch, *An introduction to projective geometry*, pp. 197-204 (1905).

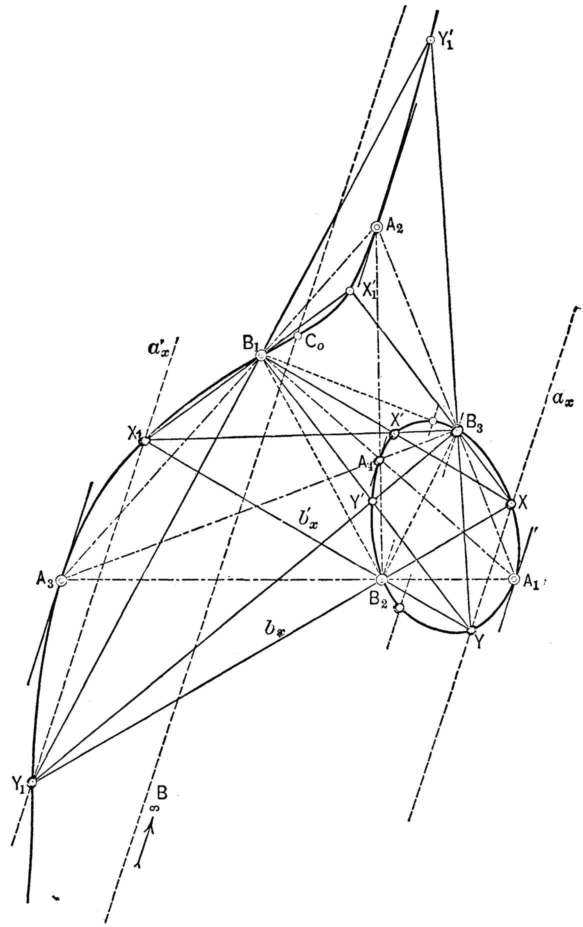


Fig. 17

the second kind on the given cubic. Thus in the involution ($B_1$, $B_3$) $X_1$ corresponds to $X$. Two involutions of the first kind, when distinct, produce an involution of the second kind. Thus the Steiner involution on $B$ sends $X$ into $Y$, the Steiner involution on $B_2$ sends $Y$ into $X_1$. Hence the product of the two central involutions is a noncentral involution $X$, $X_1$ (of the second kind).

## C. VARIOUS ALGEBRAIC CURVES

### Lantern Slides for Screen-projections

41. *Construction of Hyperelliptic Curves.*

It is well known that an hyperelliptic curve is characterized by the existence of *One* $g'_2$. The joins of the couples $(P, P')$ of this series envelope in general a curve $Km$ of class $m = n - p - 1$, if $n$ denotes the order of the hyperelliptic $Cn$ of genus $p$. Such a birational transformation $(P \leftrightarrows P')$ of the $Cn$ into itself can be incorporated in an involutorial Cremona transformation of the plane of the curve. Hence *every hyperelliptic curve may be obtained as the locus of pairs of corresponding points on the tangents of rational curves in certain involutorial Cremona transformations.*[a]

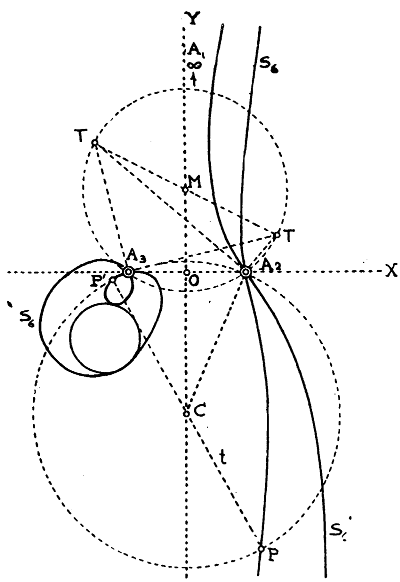


FIG. 18

A particularly simple construction for a special involutorial quadratic transformation is obtained by choosing as the invariant quadrangle.

[a]A paper by the author on the theory of such constructions will soon appear in the *Tohoku Mathematical Journal.* See also same Journal, vols. 21 (1922), pp. 310-326; 24 (1925), pp. 68-87; 25 (1925), pp. 63-76; also *Am. J. of Math.*, vol. 48 (1926), pp. 21-44.

$B$ (0, $i$), $B_2$ (0, $-$ $i$), $B_3 = I$, $B_4 = J$, where $I$ and $J$ are the circular points at infinity, Fig. 18. The pencil of conics through these points consists of circles through the intersections of the zero circles $(x - 1)^2 + y^2 = 0$ and $(x + 1)^2 + y^2 = 0$. To find the fundamental triangle $A_1A_2A_3$ of the transformation, we have at once as $A_1$ the infinite point of the $y$-axis. $A_2$ in the intersection of $B_1J$ and $B_2I$, $A_3$ that of $B_1I$ and $B_2J$. The equation of $B_1J$ is $y - i = -ix$, that of $B_2I$ $y + i = +ix$, hence their intersection (1, 0). Likewise $B_1I$ and $B_2J$ intersect at $(-1, 0)$. $A_2$ and $A_3$ coincide, therefore, with the centers of the zero-circles $C_1$ and $C_{-1}$. From this follows that all circles through $A_2$ and $A_3$ are invariant in the transformation. To construct $T'$ when any point $T$ is given, connect $T$ to $A_2$ and $A_3$; draw perpendiculars $p_2$ and $p_3$ at $A_2$ and $A_3$ to $TA_2$ and $TA_3$. Then $p_2$ and $p_3$, the polars of $T$ with respect to $C_1$ and $C_{-1}$, intersect at $T'$. Obviously, $T$, $T'$ and $A_2$ and $A_3$ lie on a circle, with $TT'$ as a diameter; hence this diameter is bisected by the $y$-axis at $M$. On every line $l$ there are, in general, two corresponding points $P$ and $P'$. To construct these, connect the point of intersection $C$ of $t$ and the $y$-axis with $A_2$ (or $A_3$), then the circle with $C$ as a center and $CA_2$ as a radius cuts $t$ in $P$ and $P'$. The analytic form of the transformation[a] is easily found as

$$x' = -x$$
$$y' = \frac{x^2 - 1}{y}$$

An interesting problem in the theory of quadratic transformations (as a matter of fact of general Cremona transformations) is the establishment of invariant algebraic curves.

In case of a conic $K_2$ in a general position this locus is an invariant sextic $S_6$ with seven double points (i. e. of genus 3) at the $A'$s and $B'$s. The sextic is bicircular and has three real double points, $A_1$ (infinite point of $y$-axis), $A_2$, and $A_3$. The construction of this sextic is extremely simple, as shown in Fig. 18. Any tangent $t$ of the circle $K_2$ cuts the $y$-axis in a point $C$. With $C$ as a center and $CA_2$ as a radius describe a circle which will cut $t$ in two points $P$ and $P'$ of the sextic $S_6$.

**42.** *The Analytic Triangle.*

The "analytic triangle" furnishes a very effective device to analyse the graphic nature of an algebraic curve in the neighborhood

[a]This is a Steinerian transformation with two pairs of conjugate complex points forming the invariant quadruple. It was not recognized as such by M. d'Ocagne. In the preceeding explanation of the Steiner transformation the notations $A$ and $B$ are interchanged.

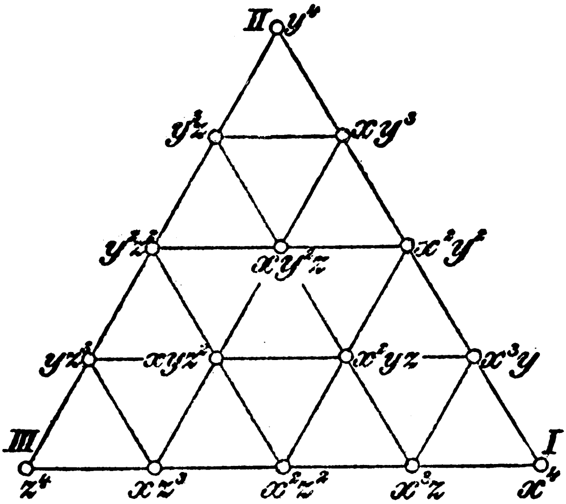


FIG. 19

of a singularity. This method, already used by Newton,[a] more fully by De Gua,[b] and later systematically developed by G. Cramer,[c] consists in writing the possible terms of an $n$-ic, in homogeneous coordinates $x$, $y$, $z$; $x^\alpha y^\beta z^\gamma$, $\alpha + \beta + \gamma = n$; at the vertices of a triangular latice-work, as indicated in Fig. 19, in case of a quartic. Those terms which are missing in the equation (coefficient $A_{\alpha\beta\gamma} = 0$) are left out in the places otherwise assigned to them in the triangle. Then we connect the extreme existing and marked terms to the left and right and below by a convex polygon. The sum of the terms of the polygonal line nearest to I, or II, or III, with the proper coefficients of the equation, set equal to zero are approximation curves of the given curve if the latter passes through one, two, or three of the vertices of the coordinate triangle.

The application of this method is shown in a number of examples in some of the following slides. For this see *Theorie der ebenen algebraischen Kurven höherer Ordnung* by Dr. H. Wieleitner, pp. 83-118.

**43.** *Assortment of Quartic Curves.*
including Quartic with 28 real double tangents.

**44.** *Rational Quartics.*

**45.** *Quintics and Sextics.*

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[a] *Enumeratio linearum tertii ordini.*—London 1704, see complete reference above.

[b] P. Sauerbeck. *Einleitung in die analytische Geometrie der höheren algebraischen Kurven. Nach den Methoden von Jean Paul De Gua De Malves.* Teubner, 1902.

[c] *Introduction à l'Analyse des Lignes courbes algébriques,* Genève, 1750.

## D. MATHEMATICAL MODELS

46. *Desargues Theorem.*

If two triangles $ABC$ and $A'B'C'$ correspond to each other such that the joins of corresponding points $A$, $A'$; $B$, $B'$; $C$, $C'$ pass through a point $S$, then the points of intersection $A_1, B_1, C_1$ of corresponding sides $BC$, $B'C'$; $CA$, $C'A'$; $AB$, $A'B'$ are collinear on a line $s$.

Conversely when two trilaterals $abc$ and $a'b'c'$ correspond to each other such that the intersections of corresponding sides $a$, $a'$; $b$, $b'$; $c$, $c'$ lie on a line $s$, then the joins $a_1$, $b_1$, $c_1$ of corresponding vertices $bc$, $b'c'$; $ca$, $c'a'$; $ab$, $a'b'$ are concurrent at a point $S$.

The model illustrating this is mounted on an aluminum base on which sets the pyramid $SABC$ made of wood with white enamel paint. The intersecting plane, made of glass cuts the pyramid at $A'B'C'$, and rests on the base along the line $s$. Projecting this from a generic point upon a generic plane, the theorem stated above results.

47. *Perspective.*

This model illustrates relation between two planes $\Sigma\ (x, y)$ and $\Sigma'\ (x', y')$ in perspective, and its interpretation in one plane ($\Sigma$ and $\Sigma'$ superposed). To the line at infinity $q$ in $\Sigma$ corresponds $q'$ (vanishing line = horizon) in $\Sigma'$. The line whose perspective $r'$ is at infinity is denoted by $r$. The planes $\Sigma$ and $\Sigma'$ intersect in the pointwise invariant line $s$. The lines $s$, $r$, $q'$ are parallel. If the center of perspective is denoted by $O$, then the distance from $O$ to $q'$ is equal to the distance of $r$ from $s$, or $\overrightarrow{oq'} = \overrightarrow{rs}$. This is still true when $\Sigma$ and $\Sigma'$ are superposed, and any such assumption is a necessary and sufficient condition for the data of a perspective.

Fig. 20 shows the construction of the perspective $K'$ of a circle $K$. Choose any line $l$ (conveniently parallel to $y$-axis) cutting $K$ in $A$ and $B$. Let $l$ cut $s$ at $S$. Draw line through $O$ parallel to $l$, cutting $q'$ at $Q'$. Then the join of $S$ and $Q'$ is the perspective $l'$ of $l$. $OA$ and $OB$ cut $l'$ at $A'$ and $B'$, two points of $K'$. As $K$ intersects $r$ in two points $T_1$ and $T_2$, with $t_1$ and $t_2$ as tangents, the perspectives of $t_1$ and $t_2$ will be tangents $t_1'$ and $t_2'$ to $K'$ at the infinite points $T_1'$ and $T_2'$, i.e., $t_1'$ and $t_2'$ are the asymptotes of $K'$. The pole $M$ of $r$ with respect to $K$ is transformed into the center $M'$ of $K'$.

Denoting the distances $\overrightarrow{Oq'}$ and $\overrightarrow{q's}$ by $a$ and $b$, the analytic form of the perspective is easily found as

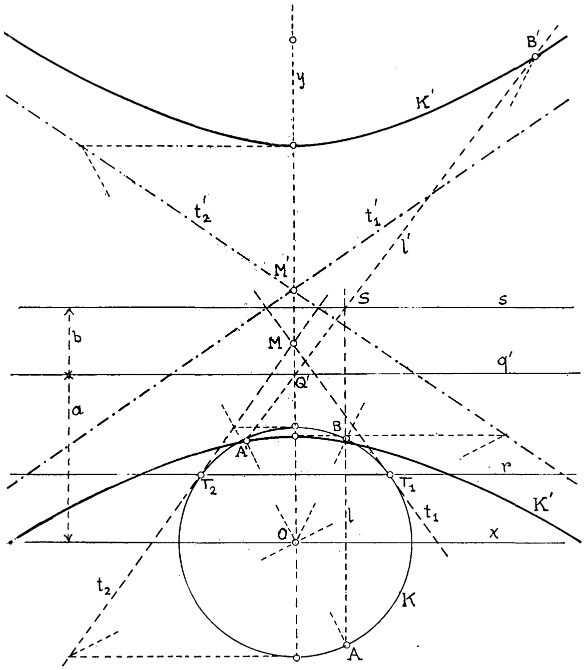


FIG. 20

$$x' = \frac{ax}{y - b}, \; y' = \frac{ay}{y - b}$$

Projecting $OSA'A$ from $U$ upon the $y$-axis we find $(OSA'A) = (OS_1A'_1A) = -a/b$ independent of the location of $A$; i.e., such a perspective has a constant cross-ratio as a characteristic. A circle $x^2 + y^2 + \alpha x + \beta y + \gamma = 0$ (by the inverse $x = bx'/(y' - a)$, $y = by'/(y' - a)$) is transformed into $b^2x'^2$ $(b^2 + \beta b + \gamma)$ $y'^2 + \alpha bx'y' - \alpha abx' - (a\beta b + 2a\gamma)$ $y' + \gamma a^2 = 0$, which may represent any curve of the second order, since there are five effective constants $a$, $b$, $\alpha$, $\beta$, $\gamma$ available, which can easily be shown to be compatible with any proper choice of constants for the transformed equation.

*Every conic may be obtained as the perspective of a circle.*

This is the reason why curves of the second order may be properly called conics. In the figure the center of the circle has been chosen at the center $O$ of perspective. It is easily proved that in this case $O$ is a focus of $K'$.

This constructive treatment of perspective is especially effective for the projection of infinite singularities. For example it can be made graphically plausible that the infinite point of the cubical parabola $y = x^3$ is a cusp, or that the infinite point of the semi-cubical parabola $y^2 = x^3$ is a flex. The construction becomes particularly simple when $a = b$. The perspective is now involutorial and the characteristic cross-ratio is $-1$.

The model has an aluminum base for $\Sigma$ and a glass-plate for $\Sigma'$ before it is made to coincide with $\Sigma$.

48. *Affine Transformation Between two Planes.*

The general affine relation between two planes $\Sigma'$ $(x', y')$ and $\Sigma$ $(x, y)$ is given by

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

with the matrix $\left\| \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\|$ of rank 2.

In case of an affine homology, or perspective, there exists a pointwise invariant line. If we choose this as the $x$-axis, then for $y = 0$, there must be $x' = x$, $y' = 0$. This is only possible when

$$x' = x + by,$$
$$y' = ey.$$

The joins of corresponding points are parallel and have the slope $(y' - y)/(x' - x) = (e - 1)/b$. If we denote two corresponding points by $P$ and $P'$ and the intersection of their join with the $x$-axis by $S$, then, as in perspective, the cross-ratio

$$(P'PS\infty) = P'S/PS = y'/y = e$$

is constant.

In the model the two planes $\Sigma$ and $\Sigma'$ are hinged along the $x$-axis and may be turned about it. The points of a circle in $\Sigma$ and its affine figure, an ellipse, in $\Sigma'$ are joined by threads which slip through the corresponding points, the threads, are the generatrices of a variable elliptical cylinder. $\Sigma$ and $\Sigma'$ are represented by aluminium-plates, Fig. 21.

49. *The Space Sextic of Genus 4.*

Models for sextics of this type are described in Series II, pp. 8-12. The object of these models is to show the six double-points of the projection of the sextic from a generic point in space upon a generic plane.

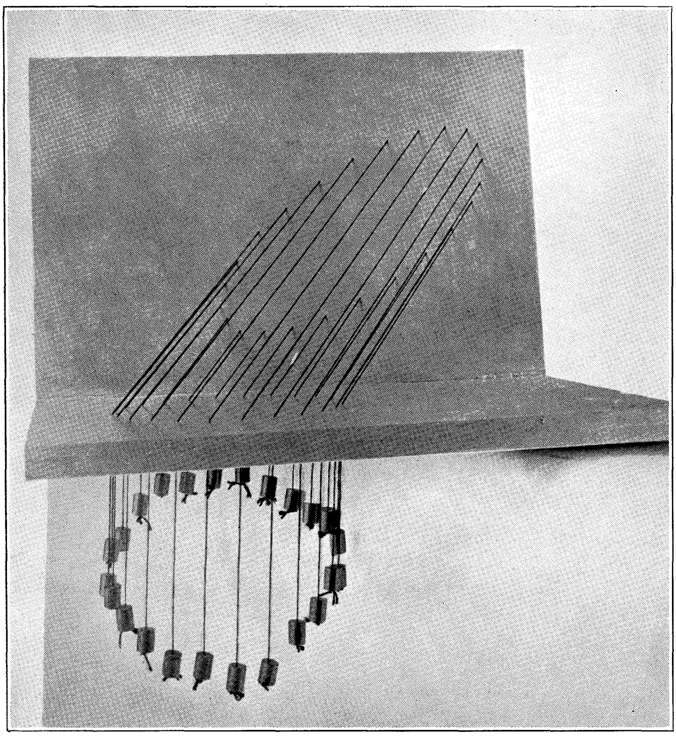

FIG. 21

As has been pointed out before the study of these sextics is of considerable importance from the standpoint of geometry as well as that of function-theory, particularly the theory of Abelian functions in connection with an algebraic curve of genus four. The canonical series is in this case a $g_{2p-2}^{p-1} = g_6^3$ cut out by the linear system of cubics $\lambda_1 \phi_1 + \lambda_2\phi_2 + \lambda_3\phi_3 + \lambda_4\phi_4 = 0$ through the six double-points. If we set the four linearly independent adjoints $\phi_i$ equal to $\rho y_i = \phi_i(x)$, $i = 1, 2, 3, 4$, the sextic in the plane is mapped into a non-singular sextic $C_6$ in $S_3$ of genus four, which may be shown to be the intersection of a quadric and a cubic surface. The canonical series on $C_6$ is now cut out by the planes.

$$\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \lambda_4 y_4 = 0.$$

Among the sextuples of the series those consisting of three couples of coincidences are of particular importance. Their number is obviously equal to the number of tritangent planes of $C_6$.

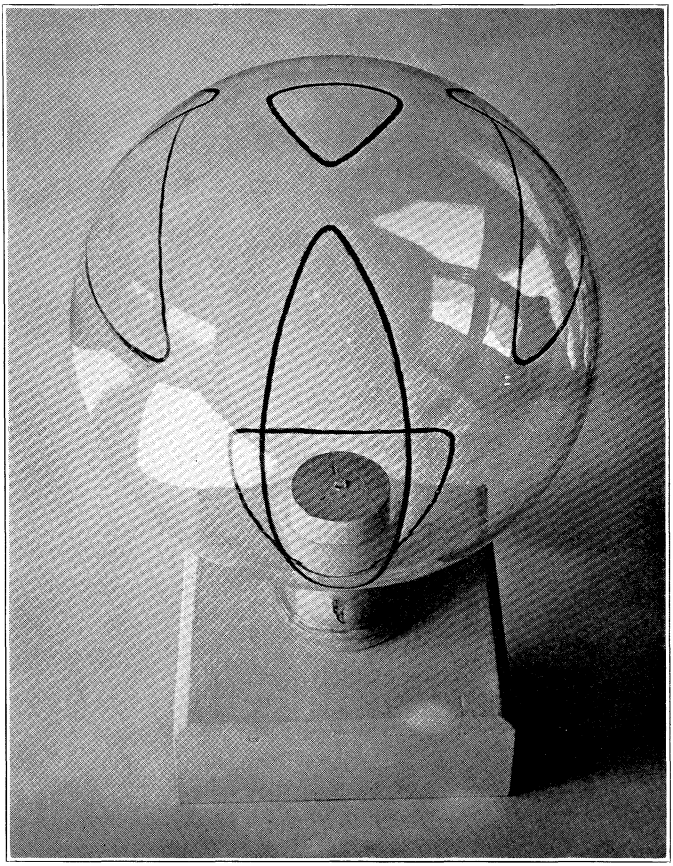

FIG. 22

One method to gain some insight into the configuration of these tritangent planes is by the study of the Riemamian theta functions connected with the problem. It is thus found that there are $2^{p-1}(2^p - 1) = 2^3(2^4 - 1) = 120$ point-groups of the canonical series, each consisting of three couples of coincidences. Hence the general sextic $C_6$ of genus four in $S_3$ has 120 tritangent planes.

The question may be asked, is it possible that all 120 tritangent planes may be real? Model No. 49 has been designed to show that this is possible. The sextic is obtained as the intersection of a cubic cone whose base, Fig. 23, in the plane $x_4 = 0$ may be written as

$$x_1x_2x_3 - (a_1x_1 + a_2x_2 + a_3x_3)^3 = 0,$$

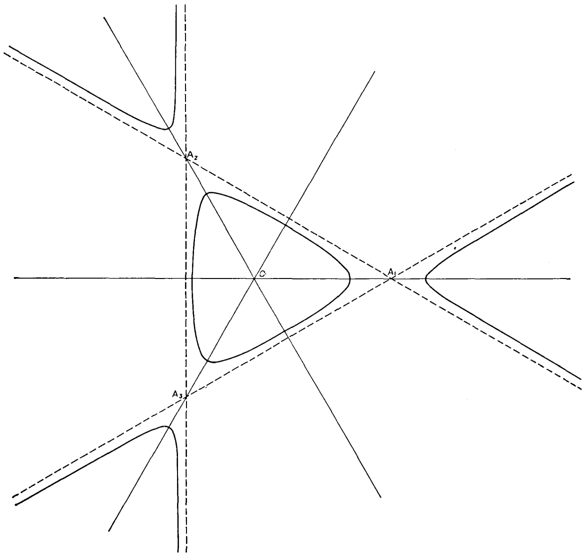


FIG. 23

so that the sides of the coordinate triangle are flex-tangents, and a sphere. To obtain constructive symmetry, $\Sigma a_i\, x_i = 0$ has been chosen as the line at infinity; moreover the collineation has been arranged so that the inflexional asymptotes form an equilateral triangle. The resulting space sextic consists of 5 ovals, the maximum number possible. It has threefold symmetry with respect to three planes through a vertical diameter of the sphere, dividing this into 6 sections of 120° each. Denoting the lateral ovals by $D$ and $E$, there are first $\binom{5}{3}\cdot 8 = 80$ tritangent-planes, with single contacts with the ovals. Each of the ovals $A$, $B$, $C$ is touched twice by 6 tritangent planes, which gives 18 of this type. For each $D$ and $E$, there are $3.3 = 9$ tritangent-planes with double-contact, hence $2.9 = 18$ of this type.

Moreover there is a plane touching $D$ in three points, likewise one for $E$. Lastly there are two horizontal tritangent-planes touching $A$, $B$, $C$ from above and from below. Thus we count altogether. $80 + 18 + 18 + 4 = 120$ tritangent-planes. It must be remembered that this is no proof for the reality of all tritangent-planes, it merely shows the possibility constructively.

**50.** *Surface of Constant Curvature.*

Models of surface of rotation of constant positive curvature and of sphere upon which it can be developed. Made of wood. A de-

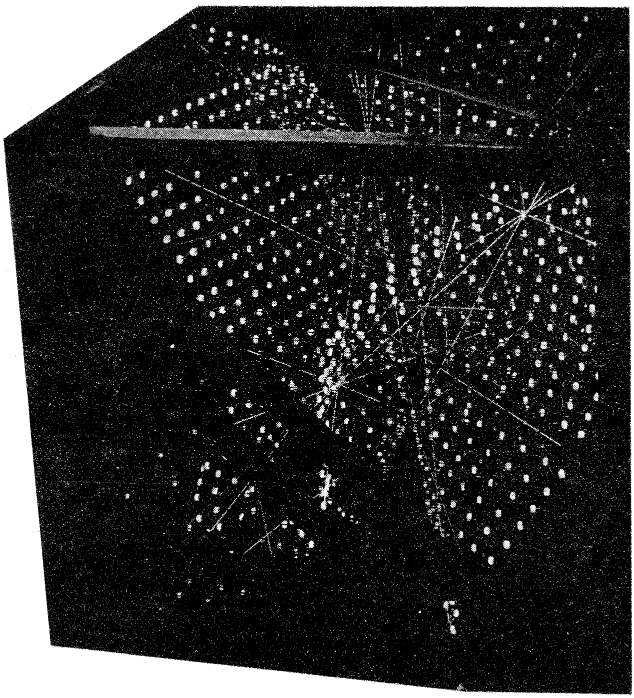

Fig. 24

formable hollow thin brass-shell shows how these surfaces are mutually developable.

51. *Weddle Surface* (By W. L. Moore).

The surface belongs to the class of determinant surfaces, this one being the Jacobian of the four quadrics

$$x^2 + y^2 + z^2 - (x + y + z) = 0,$$
$$xy - 4xz + 3yz = 0,$$
$$3(x^2 + xy - x) - (z^2 + yz - z) - 2xz = 0,$$
and $$5(y^2 + xy - y) - 2(z^2 + xz - z) - 3yz = 0,$$
on the six points

$$A_1(0, 0, 0),\ A_2(1, 0, 0),\ A_3(0, 0, 1),\ A_4(0, 1, 0),\ A_5(1, 1, 1),$$

and $A_6(1/3, 2/3, -1/3)$. Geometrically it is the locus of vertices of quadric cones through the six points.

Its equation is

$$\begin{vmatrix} 2x-1 & 2y-1 & 2z-1 & -x-y-z \\ y-4z & x+3z & -4x+3y & 0 \\ 5y-2z & 5x+10y-3z-5 & -2x-3y-4z & 2 & -5y+2z \\ 6x+3y-2z-3 & 3x-z & -2x-y-2z+1 & -3x+z \end{vmatrix} = 0$$

or

$$(x+y+z-1)\ [4x^2y - 4xy^2 + 2x^2z - 2xz^2 + 2yz^2 \quad -2y^2z + xy -4xz + 3yz] \quad -2xyz\ (3x - 2y - z) = 0.$$

The model shows the six nodes, the fifteen lines joining the nodes in pairs, the ten lines which are the intersections of the pairs of planes through the nodes, and the space cubic through the six nodes. This cubic is in any case an asymptotic line of the corresponding Weddle Surface. The plane sections are represented by beads outlining the quartic curves thus giving several systems of quartic curves on the surface.