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MATHEMATICAL MODELS

III SERIES

BY

ARNOLD EMCH

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PREFACE

The mathematical models listed and described below have been planned and designed in the mathematical laboratory of the University of Illinois since the publication[1] of the first series in 1921 and the second series in 1923. As has been stated in the first two series, the purpose of these constructions is to represent certain features of mathematical instruction and research by adequate models, mechanisms, or graphs, which are not available in the market.

In the second series four models are described which illustrate the connection between the apparent double-points and the genus in case of an important class of space-curves, those sextics, which are obtained as intersections of quadric and cubic surfaces. In the new series models are presented which exhibit similar features in case of quartics of the first kind.

The theory of surfaces is represented by a cubic, a quartic, and a quintic. The latter is a special example of a class of generalized cyclides, whose properties were studied by Dr. H. P. Pettit. References to these surfaces will be found in the text.

Particular interest may be attached to the model, illustrating geometrically the properties of the symmetric substitution group of order 24. It is believed to be the first of its kind in existence.

For those interested in models as listed in series I, II, and III, arrangements can be made with private firms for the manufacturing and sale of duplicates.

For further information, especially concerning the procuring of duplicates, apply to Arnold Emch, Associate Professor of Mathematics, University of Illinois.

[1]University of Illinois Bulletin, Vol. XVIII, No. 12, formally dated Nov. 22, 1920; and Vol. XX, No. 42, June 18, 1923.

MATHEMATICAL MODELS

III SERIES[1]

Intersection of Quadrics

As is well known two quadrics in a general position intersect in a quartic C_4 of the first kind, i.e., its projection C'_4 from a generic point upon a generic plane has two double points (apparent double points for the space-quartic), and is thus of genus 1. To construct

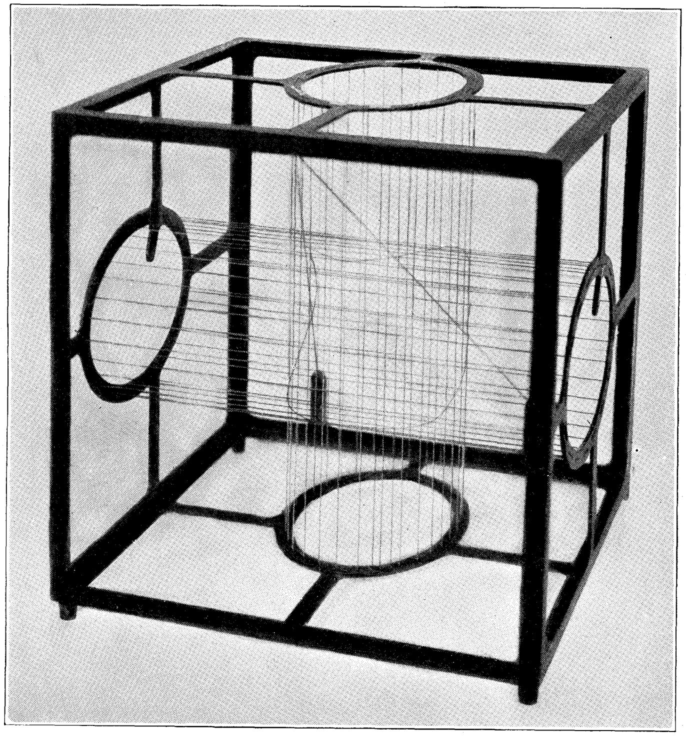

Fig. 1

[1]The last number of series II was 25. In this, as in prospective series, the models will be numbered continuously, so that in any inquiry concerning these models it will suffice to state the corresponding number.

these double points, let O be the center of projection and Q_1 and Q_2 the two quadrics. The polar planes of O with respect to Q_1 and Q_2 intersect in a line d whose projection d' contains the two double points of C'_4, as is easily proved by projective methods.

26. Quartic in space with one effective double-point.

As the pencil of quadrics through a quartic, determined by two quadrics Q_1 and Q_2 has, in general, four quadrics with double points, i.e., cones or cylinders, which may be real or imaginary, we do not restrict the generality of the result projectively, by assuming Q_1 and Q_2 as cylinders. In this model the cylinders touch in a point so that the quartic C_4 has an effective double point. As there are moreover two apparent double points, the C_4 is rational. The construction of the apparent double points is indicated by yellow silk strings, Fig. 1.

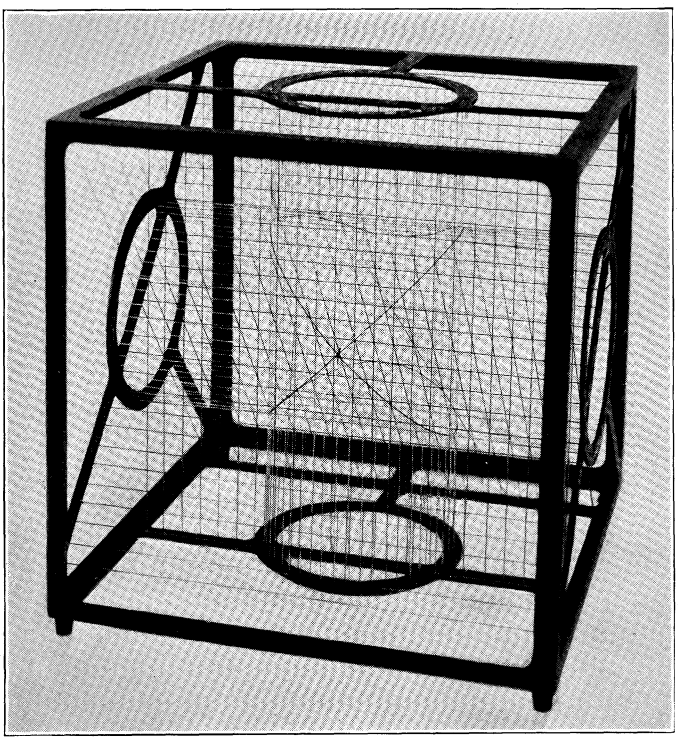

Fig. 2

27. Quartic of genus 1.

The two cylinders Q_1 and Q_2 are put in such a position that there is partial penetration, so that the quartic C_4 consists of one branch only. The curve is closed. As in No. 26, the construction of the two apparent double points is represented by yellow strings.

In both models, cylinders, polar-planes, etc., are shown in different colors, Fig. 2.

Surfaces Generated by Certain Systems of Circles

In a paper on incidences of straight lines and plane algebraic curves, and surfaces generated by them,[1] one of the problems was the description of surfaces by systems of plane algebraic curves determined by the intersections of their planes with certain fixed lines and curves.

The following two models are examples of simple special cases of such surfaces, obtained by assuming three lines l, g, h and another fixed line s and the pencil of planes through s. Every plane of the pencil cuts l, g, h in three points which determine a circle. The locus of these circles is, in general, a quartic surface.

28. Quartic Surface.

Let the three lines be defined by

$$l \quad \begin{cases} x = a_1 z + b_1 \\ y = c_1 z + d_1 \end{cases}$$

$$g \quad \begin{cases} x = a_2 z + b_2 \\ y = c_2 z + d_2 \end{cases}$$

$$z \quad \begin{cases} x = a_3 z + b_3 \\ y = c_3 z + d_3, \end{cases}$$

and assume s as the infinite line of the z-plane, then the equation of the quartic may be written in the form

$$\begin{vmatrix} x^2 + y^2 & , & x & , & y & , 1 \\ (a_1^2 + c_1^2)z^2 + 2(a_1 b_1 + c_1 d_1)z + b_1 d_1 & , & a_1 z + b_1 & , & c_1 z + d_1 & , 1 \\ (a_2^2 + c_2^2)z^2 + 2(a_2 b_2 + c_2 d_2)z + b_2 d_2 & , & a_2 z + b_2 & , & c_2 z + d_2 & , 1 \\ (a_3^2 + c_3^2)z^2 + 2(a_3 b_3 + c_3 d_3)z + b_3 d_3 & , & a_3 z + b_3 & , & c_3 z + d_3 & , 1 \end{vmatrix} = 1,$$

[1]The American Journal of Mathematics, Vol. XLIV (1922), pp. 12–19.

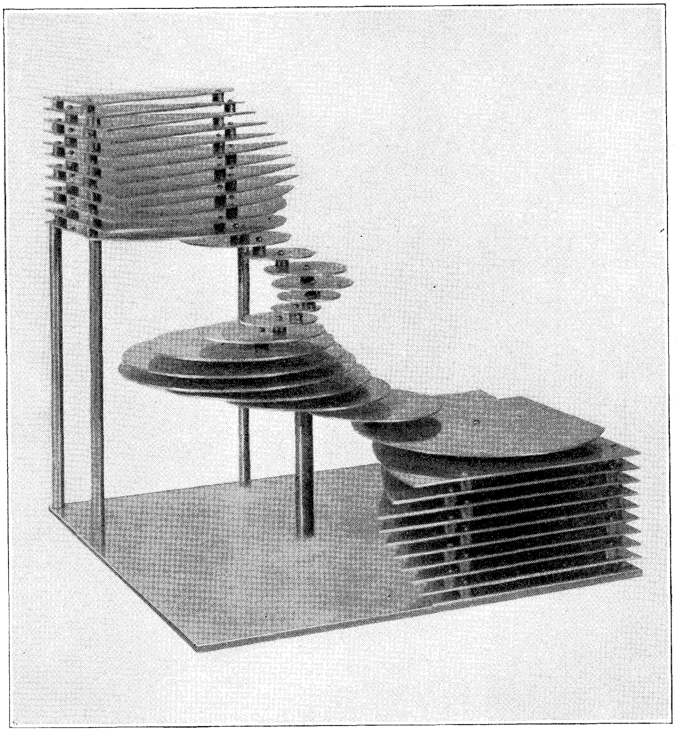

FIG. 3

which reduces to an equation of the fourth degree. Every plane $z = k$ cuts this surface in a circle and must therefore cut the surface in another conic. Making the equation homogeneous, it is easily seen that the line s is a nodal line which absorbs that second conic. There are two other lines on the quartic, namely the two possible transversals p and q of l, g, h, and s. Thus there are altogether 5 lines and a nodal line on the surface, Fig. 3.

29. Cubic Surface.

The method of generating this surface is the same as in No. 28. The three lines l, g, h, of which the first two intersect at the origin, are defined by

$$l \begin{cases} x = z \\ y = z \end{cases} \qquad g \begin{cases} x = z \\ y = -z \end{cases} \qquad h \begin{cases} x = 1 \\ y = 0 \end{cases}$$, while

s is the same as before. The quartic of No. 28 degenerates into the z-plane and the cubic:

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 2z^2 & z & z & 1 \\ 2z^2 & z & -z & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} = 0,$$

which reduces to

$$(x^2+y^2)\,(z-1) - x(2z^2-1) + 2z^2 - z = 0.$$

The y-plane touches the cubic along h. The intersection k of the planes $x = 1, z = 1$ lies on the cubic, and the plane $z = 1$ touches the cubic along s, the infinite line of the z-plane. Thus the cubic has 7 real lines, of which 4 are absorbed by h and s.

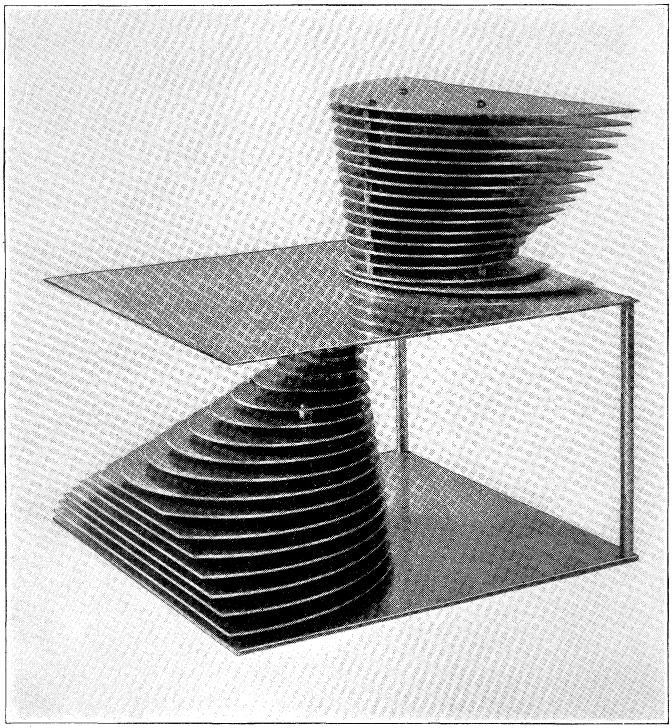

FIG. 4

The cubic may be put in the form

$$\left\{x - \frac{2z^2 - 1}{2(z - 1)}\right\} + y^2 = \left\{\frac{2z^2 - 1}{2(z - 1)}\right\}^2 - \frac{2z^2 - 1}{2(z - 1)} + 1 ,$$

which shows that the circles on the cubic arrange themselves into couples of equal circles (2 values of z) such that their projections on the z-plane are equal, Fig. 4.

The intersection of the y-plane with the cubic is the hyperbola

$$2z^2 - xz + x - z = 0 \quad ,$$

with the asymptotes $z = 1$ and $z = \frac{1}{2}x - \frac{1}{2}$;
and the line $x = 1$.

In both models the circular sections are represented by aluminum disks. They are also reproduced by plaster-casts.

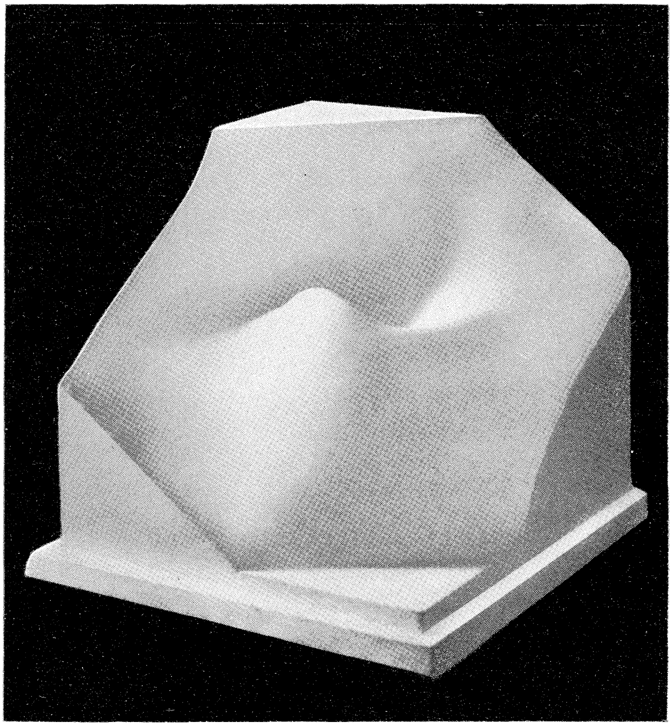

FIG. 5

30. Quintic cyclide.[1] (By Dr. H. P. Pettit)

The surface is one of the few anallagmatic quintic cyclides, this one being anallagmatic with respect to eight spheres. It is projectively generated by the tangent planes of a cone of second order and a pencil of concentric spheres

$$x\lambda^2 + y\lambda + z = 0,$$
$$P - \lambda Q = 0,$$

where $P = x^2 + y^2 + z^2 - r^2, \quad Q = x^2 + y^2 + z^2 + r^2,$

and has the form

$$x(x^2 + y^2 + z^2 - r^2)^2 + y(x^2 + y^2 + z^2 - r^2)(x^2 + y^2 + z^2 + r^2) + z(x^2 + y^2 + z^2 + r^2)^2 = 0.$$

The vertex of the double tangent cone and the center of the pencil of spheres coincide at the origin. The surface has one infinite set of circular generators and is symmetrical with respect to the origin. The section by the xy-plane is a quintic composed of a cubic and a circle of radius r. The section by the xz-plane is a proper quintic. The section by the yz-plane is a degenerate quintic composed of a cubic and an imaginary circle. The topography in the neighborhood of the xy-plane shows the change from the degenerate quintic to the proper quintic with the loop and serpentine, Fig. 5.

31. Model of the symmetric substitution group of order 24 (G_{24}).

The symmetric substitution group of n elements of order $n!$ may be represented geometrically in a projective space of $n-1$ dimensions and thus lends to important properties of certain geometric forms. The author has recently[2] considered some of these applications, principally with reference to the G_6 and G_{24}. The four elements, numbers, x_1, x_2, x_3, x_4 of the G_{24} are chosen as projective coordinates of a point in 3-space.

The model illustrates the theorem:

Any set of 24 points of the G_{24} lies on 16 conics of a quadric whose planes by four pass through the four lines s_i cut out from the unit-plane e by the coordinate-planes $x_i = 0$. The points of the group lie two by two on 72 lines which in sets of 12 pass through the six vertices E_{ik} of

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[1] A general cyclide with special reference to the quintic cyclide. The Tohoku Mathematical Journal, Vol. 23 (1923), pp. 1–25.

[2] The American Journal of Mathematics, Vol. XLV (1923), pp. 192–207.

the quadrilateral $s_1s_2s_3s_4$. *24 points of the* G_{24} *form 6 involutions with the* E_{ik}*'s as centers and the 6 planes through the edges of the coordinate-tetrahedron and the unit-point, taken in the proper order, as axial planes. All quadrics of the* G_{24} *form a pencil and touch each other and the cone* $\Sigma(x_i - x_k)^2 = 0$ *along its intersection with e.*

For convenience, the construction has been carried out in Carterian space, so that when $x_h x_i x_k x_l$ denotes any of the substitutions, the point will have the coordinates $x = x_h/x_l$, $y = x_i/x_l$, $z = x_k/x_l$. In the model the starting point is (1234), or $x = \frac{1}{4}$, $y = \frac{2}{4}$, $z = \frac{3}{4}$. The scale is then throughout multiplied by 72, so that this point is now (18, 36, 54). The six sets of 12 lines each are shown in different colors, Fig. 6.

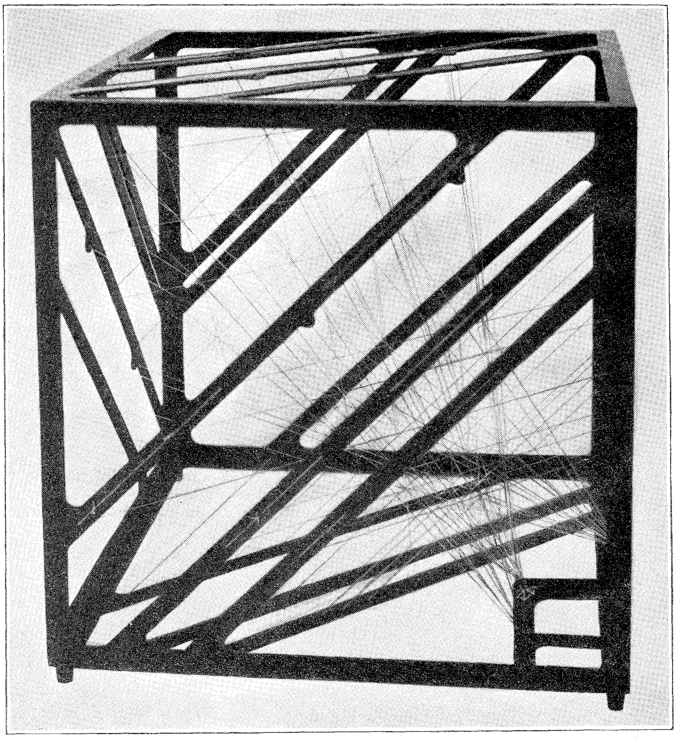

FIG. 6