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MATHEMATICAL MODELS

BY

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MATHEMATICAL MODELS

I SERIES

Introduction

The mathematical models listed below have been designed and constructed in the mathematical department of the University of Illinois and are the first results of an effort to represent certain desirable features of mathematical instruction and research by adequate models, mechanisms, or graphs, when they are not available otherwise.

In this as well as in some prospective lists only such problems will be considered which have not found the same constructive treatment elsewhere and of which there are no models in the market. Among the projects considered for future work are designs for stereoscopic and ordinary lantern slides, cinematographic films and charts of mathematical figures.

In connection with this work a course in Mathematical Graphics and Modelling is under consideration. This would be for students interested in this work or wishing to supplement their customary mathematical training by acquiring the technique and various methods of actual graphic representation and model-making.

Moreover suggestions of problems along this line will be gladly accepted and investigated with reference to their constructive possibilities.

For those interested in such models, it will be possible to make arrangements with private firms for the reproduction and sale of the following models at prices which will be quoted upon application.

Orders for any of these models will be filled by these firms as fast as they can be manufactured under present conditions.

Quadric

1. *Variable String Model of Hyperbolic Paraboloid.*

This model is independent of the usual frame upon which the strings are attached. It consists of two metal bars with handles. The bars are perforated by a set of holes, which, imagined as points, are projectively related. The two sets are arranged such that they may be similar, or congruent, so that the regulus represented by the strings connecting corresponding points is an hyperbolic paraboloid. The ends of the strings are fixed to the points of one set, while the strings, with weights at the other ends, may slide freely through the holes of the second set, so that by displacing the two bars arbitrarily in space, an infinite number of such surfaces may be represented.

String Models of a Certain Class of Rational Algebraic Ruled Surfaces

Besides the hyperboloid of rotation of one sheet and the class of helicoids, very little is known of cinematically generated ruled surfaces. The class here considered is obtained as follows: A fixed point M of a generatrix g moves in a circle C_2 with a certain angular velocity w. Let C_1 be the line through the center O, and perpendicular to the plane of the circle C_2, coincide with the z-axis of a cartesian system. In a second relative motion let g rotate about M in a plane through g and C_1, with an angular velocity of $p/q\,w$ where p and q are relative prime integers. Under these conditions g generates a rational algebraic ruled surface of order $2(p+q)$, or $p+q$, according as q is odd or even.

A full investigation* of these surfaces has been made and will be added in circular form to the set of four models representing simple cases of these surfaces.

The model frames are made of brass and covered by a black lacquer.

2. Cubic, $p=1$, $q=2$.

Both C_1 and C_2 are single lines of the surface.
Cartesian equation:

$$[(x-a)^2 - (z-y)^2]\,y - 2x(x-a)(z-y) = 0.$$

Parametric equation:

$$x = s\,\frac{1-t^2}{1+t^2}, \quad y = s^2\frac{t}{1+t^2}, \quad z = \frac{s-a}{t},$$

with s and t as parameters and a as the radius of C_2. This surface is unifacial and has the straight line $x=a$, $y=at$, $z=at$, as a double curve.

3. Quartic, $p=1$, $q=1$.

Cartesian equation:

$$(x^2+y^2)\,x^2 - (ax+yz)^2 = 0.$$

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*On a certain class of rational ruled surfaces, The American Journal of Mathematics, Vol XLII, pp. 189-210 [1920.]

This is the *6th* species in the classification of ruled quartics by Cayley, and the *5th* species in that of Cremona. The surface has C_1 as a double line which must be counted twice. Moreover there is another double line ($x = 0$) in the xy-plane.

4. *Quintic,* $p=3$, $q=2$.

Cartesian equation:

$$(y^3 - 3x^2y)\,z^2 + (2x^4 + 4x^2y^2 + 2y^4 - 2ax^3 + 6axy^2)\,z + (x^2 + y^2 - a^2)(y^3 - 3x^2y) = 0.$$

C_1 is in a triple line of the surface and must be counted as three double-lines. The surface has the twisted cubic

$$x = \frac{a(1+t^2)}{1-3t^2}, \quad y = \frac{at(1-t^2)}{1-3t^2}, \quad z = \frac{a(3t+t^3)}{1-3t^2}$$

as a double curve. In Schwarz's classification of quintic ruled surfaces this is marked with III A.

5. *Sextic,* $p = 2$, $q = 1$.

Cartesian equation:

$$(x^2 + y^2)(x^2 - y^2)^2 - [2xyz - a(x^2 - y^2)]^2 = 0.$$

C_1 is a quadruple line of the surface and must be counted as eight double lines. There are two double generatrices $x \pm y = 0$ in the xy-plane.

Tangent Surfaces of a Rectilinear (2,2)-Congruence

The joins of corresponding points z' and z of two complex planes E' and E in space, related by

$$z' = \frac{ax+b}{cz+d}, \quad ab-bc \neq 0.$$

form a congruence. The elements of such a congruence touch a certain quartic surfaces Q.

Plaster cast models of two such surfaces have been constructed:

6. Q_1 *with a system of elliptic sections.* (C. N. Stokes)

Cartesian equation:

$$(x^2 + y^2 + z^2)(x^2 + 4y^2) - 36(x^2 + y^2) = 0$$

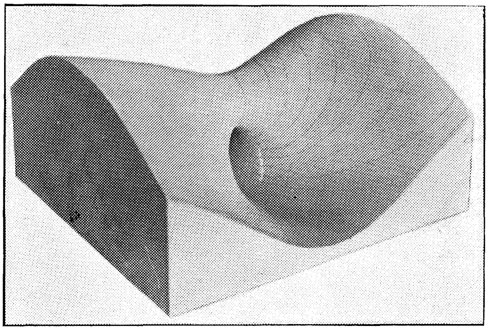
7. *Q_2 with a system of hyperbolic sections.* (C. N. Stokes)
Cartesian equation:

$$16(x^2 + y^2)(x^2 + 4y^2 - 4) + (x^2 + 4y^2)[7(x^2 + 4y^2) - 16z^2 - 28] = 0.$$

Projective Generation of Surfaces

8. *Cubic Surface (Cyclide)*

Generated by a pencil of planes, as



$$3ax - \lambda z = 0.$$

and a projective pencil of spheres

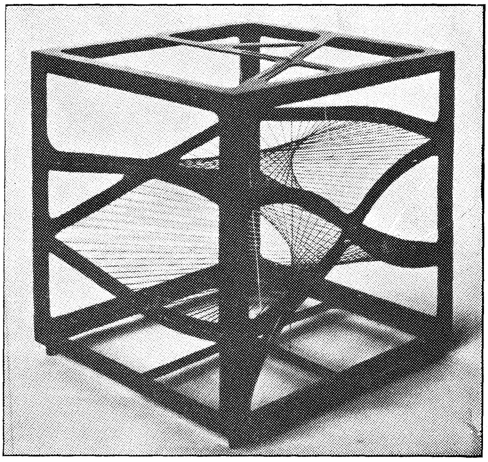
$$x^2 + y^2 - (z-3a)^2 - 2\lambda z = 0.$$

This is a plaster cast model of a cubic cyclide and shows the circles as intersections of corresponding planes and spheres, and also the 7 real lines which are on the surface in this case.

9. *Quartic ruled surface.*

Generated by the projectively related planes of two cones of the second class. The vertices of the two cones are at (0, 0, 0), and $(-2a, 0, 0)$ of a Cartesian system, and the tangent planes make angles of 45° with the xy-plane. The equation of the qaurtic is

$$(a + x)^2 z^2 - (a^2 - z^2) y^2 = 0.$$



It arises in connection with the study of cyclides. If we consider the quadric locus of the centers of all double tangent spheres associated with one of the five systems by which the cyclide may be generated, the centers of all circles of the system on the cyclide lie on two quartic curves cut out from the quadric by the quartic ruled surface.

Cauchy Surfaces

If we consider a function of a complex variable $w = f(\zeta) = u + iv$, in which $\zeta = x + iy$, and at every point (x, y) erect a perpendicular to the complex ζ – plane equal to the value of $u^2 + v^2$ at each point, and denote this value by z, we obtain an equation $F(x, y, z) = 0$ which defines a so called Cauchy surface This surface is the locus of the endpoints of those perpendiculars.

10. Surface for $w = \dfrac{\zeta - 1}{\zeta + 1}$

Cartesian equation:

$$z = \frac{(x^2 + y^2 - 1)^2 + 4y^2}{[(x+1)^2 - y^2]^2}$$

a quintic.

11. Surface for $w = \zeta^3 - 1.$

Cartesian equation:

$$z = (x^2 + y^2)^3 - 2x^3 + 6xy^2 + 1,$$

which is a sextic.

Plane Sections of Torus

The torus belongs to the class of surfaces known as cyclides and has the isotropic circle of space as a double-curve. An ordinary plane section of a cyclide is therefore a bicircular quartic. When the plane touches the surface in one point, the quartic has three double points and is therefore rational. A double tangent plane cuts the surface in a quartic with four double points. This quartic must therefore necessarily degenerate into two conics. But as the quartic is bicircular, these conics contain the circular points and are therefore circles.

The sections in case of a single and a double tangent plane are shown in two models made of wood and brass-plates indicating the sections.

12. Section of Torus and Single Tangent Plane. Rational quartic.

13. Section of Torus and Double Tangent Plane. Quartic degenerating into two circles.

Cellular Division of Space

14. System of rhombohedrons and dodecarhombohedrons.

This model shows how a hexagonal space may be packed by dodecarhombohedrons and rhombohedrons of a definite form and has its origin in the constructive verification of an answer to a dentist concerning certain cellular structures of space. See American Mathematical Monthly, Vol. XXV, pp. 128-131, March 1918.

The model is made of wood and the individual cells are painted white.

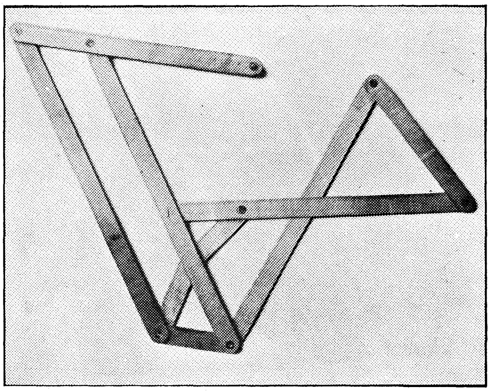
15. Closed unilateral surface.

In the sense of analysis situs this surface is equivalent with the projective plane. The model consists of a spherical region, which passes over into a ruled surface (made of strings) along a twisted quartic curve.

Linkages

Linkages are articulated systems of bars which are connected by hinges or joints, and are deformable. In a pure linkage no sliding motion is admitted. Kempe, Darbonx, and others, have proved that all algebraic relations between a finite number of real or complex variables may be realized by linkages in the plane or in space. Thus, all algebraic curves may be generated by such linkages.

16. Hebbert's Cardioidograph.



This is a comparatively very simple and elegant linkage by which the ordinary cardioid may be described. See American Mathematical Monthly, Vol. XXII, pp. 12-13 (1915).

17. Mechanism illustrating cinematical description of certain ruled surfaces. (Hebbert)

18. Cinematographic film of Poncelet polygon.

Continuously appearing movement of a triangle remaining inscribed and circumscribed to two fixed circles respectively.

For further information apply to Arnold Emch, Associate Professor of Mathematics, University of Illinois.

Urbana, Illinois.
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